

# On the bubbling set of the Yang-Mills flow on a compact Kähler manifold

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## Abstract

We study the Yang-Mills flow on a holomorphic vector bundle  $E$  over a compact Kähler manifold  $X$ . We show that the curvature of the evolved connection is uniformly bounded away from an analytic subvariety determined by the Harder-Narasimhan-Seshadri filtration of  $E$ . As a consequence, we prove that the bubbling set is a unique, holomorphic subvariety of  $X$  depending only on the isomorphism class of  $E$ . This verifies a conjecture of Hong-Tian in Kähler case.

## 1 Introduction

A current theme in complex differential geometry is the connection between existence of canonical geometric structures and algebraic stability in the sense of geometric invariant theory (GIT). This theme is in part motivated by the famous theorem of Donaldson-Uhlenbeck-Yau, which states that the existence of a Hermitian-Einstein connection on an indecomposable holomorphic vector bundle  $E$  over a Kähler manifold  $(X, \omega)$  is equivalent to the stability of  $E$  in the sense of Mumford-Takemoto [4, 27]. This theorem was first observed by Narasimhan and Seshadri [13] in the case of complex curves, by Donaldson for algebraic surfaces [4], and by Uhlenbeck and Yau [27] in arbitrary dimension. A heat flow approach to the existence of Hermitian-Einstein connections, related to the Yang-Mills flow, was introduced by Donaldson in [4]. This approach has been extended to several more general settings [1, 5, 17, 18].

A related area of research in complex differential geometry which has garnered a great deal of interest recently is the connection between the limiting properties of various geometric heat equations and algebraic geometry. Most notably, a recent program of Song-Tian aims to relate the singularities of Kähler-Ricci flow on projective varieties to the minimal model program in algebraic geometry. Several striking results have been obtained in this direction by a number of authors; see, for instance [19, 20, 21, 22, 23, 24, 25, 26], and the references therein.

The current work is an advance in both of these directions in the setting of the Yang-Mills flow. It is by now well known that the Yang-Mills flow on a vector bundle  $E$  converges to a Hermitian-Einstein connection if and only if  $E$  is stable, [4, 5, 18]. As a result, it is natural to study the limiting properties of the Yang-Mills flow when  $E$  is not stable. When  $\dim_{\mathbb{C}} X = 2$ , this problem was studied extensively by Daskalopoulos and Wentworth [2, 3]. They found that many limiting properties of the Yang-Mills flow are determined by the algebraic structure of  $E$ . More precisely, Daskalopoulos and Wentworth show that away from an analytic bubbling set the Yang-Mills flow converges to a Yang-Mills connection on

the direct sum of the stable quotients of the graded Harder-Narasimhan-Seshadri filtration [2]. We denote this direct sum of stable quotients by  $Gr^{hns}(E)$ . In the later paper [3], Daskalopoulos and Wentworth show that the analytic bubbling set is precisely equal to the set where the stalk of the torsion-free sheaf  $Gr^{hns}(E)$  fails to be a free module. This provides a remarkable and deep connection between the limiting behaviour of the Yang-Mills flow and the GIT of the bundle  $E$  on which we briefly elaborate. In a recent paper, Gómez, Sols and Zamora [6] showed that an unstable torsion-free sheaf on a projective variety gives rise to a GIT-unstable point in a certain Quot scheme. To this unstable point, the theory of Kempf-Ness associates a maximally destabilizing 1-parameter subgroup which in turn induces a filtration of  $E$  by weight spaces. Gómez, Sols and Zamora show that this maximally destabilizing filtration is precisely the Harder-Narasimhan filtration of  $E$ .

For Kähler manifolds of arbitrary dimension, the second author partially generalized the work of Daskalopoulos and Wentworth, proving the limiting reflexive sheaf along the Yang-Mills flow is in fact isomorphic to  $Gr^{hns}(E)$  [10, 11]. In this paper we carry this generalization further by proving that the analytic bubbling set along the flow is precisely where  $Gr^{hns}(E)$  fails to be locally free. More precisely, given a subsequence of times  $t_j$  along the Yang-Mills flow, following Hong-Tian [8] we define the analytic singular set (or bubbling set) to be:

$$Z_{an} = \bigcap_{r>0} \{x \in X \mid \liminf_{j \rightarrow \infty} r^{4-2n} \int_{B_r(x)} |F_A(t_j)|^2 \omega^n \geq \varepsilon\}.$$

A large part of Hong and Tian's paper [8] is dedicated to proving certain properties of  $Z_{an}$ , however uniqueness and dependence on the choice of subsequence  $t_j$  is left open. They do show that along such a subsequence the Yang-Mills flow converges smoothly on  $X \setminus Z_{an}$ , modulo gauge transformations, to a Yang-Mills connection on a limit bundle  $E_\infty$  on  $X \setminus Z_{an}$ . In [1] Bando and Siu prove this bundle extends to all of  $X$  as a reflexive sheaf  $\hat{E}_\infty$ , and the second author proves in [11] that  $\hat{E}_\infty \cong Gr^{hns}(E)^{**}$ . Since  $E_\infty$  is locally free on  $X \setminus Z_{an}$ , the stalk of  $Gr^{hns}(E)$  must be free away from  $Z_{an}$ . Denote the set where  $Gr^{hns}(E)$  fails to be free by  $Z_{alg}$ ; we refer to this set as the algebraic singular set. Then a corollary of the main result of [11] is that  $Z_{alg} \subseteq Z_{an}$ . The goal of this paper is to prove the reverse inclusion,  $Z_{an} \subseteq Z_{alg}$ , showing that bubbles do not form away from the algebraic singular set.

Before stating the main theorems, let us recall some basic definitions. Let  $E$  be an indecomposable holomorphic vector bundle over a compact Kähler manifold  $(X, \omega)$ . One can always find a Harder-Narasimhan-Seshadri filtration,

$$0 = S^0 \subset S^1 \subset S^2 \subset \dots \subset S^p = E, \quad (1.1)$$

defined to have torsion free, stable quotients  $Q^i = S^i/S^{i-1}$ . Such a filtration may not be unique, however, the direct sum of stable quotients  $Gr^{hns}(E) := \bigoplus_i Q^i$ , is uniquely determined by the isomorphism class of  $E$ . It follows that the the *algebraic singular set* of  $E$ , given explicitly by

$$Z_{alg} := \{x \in X \mid Gr^{hns}(E)_x \text{ is not free}\},$$

is uniquely determined by the isomorphism class of  $E$ . If  $E$  is not stable it does not admit a Hermitian-Einstein connection, so we do not expect the Yang-Mills flow to converge smoothly to a limiting Yang-Mills connection. In particular, we expect that bubbles should form in the limit as  $t \rightarrow \infty$ . The following theorem gives optimal control of the bubbling set.

**Theorem 1.** *Let  $A(t_j)$  be any sequence of connections along the Yang-Mills flow. Then on  $X$ , the analytic singular set is the same as the algebraic singular set of  $E$ :*

$$Z_{an} = Z_{alg}.$$

This theorem implies that the bubbling set of the Yang-Mills flow is a unique, holomorphic subvariety of  $X$  which is independent of the subsequence chosen along the flow. As a result, this theorem verifies a conjecture of Hong-Tian [8] in the Kähler setting. In fact, we prove the slightly stronger theorem:

**Theorem 2.** *Let  $A(t)$  be a family of connections evolving along the Yang-Mills flow. Then for any compact set  $K \subset X \setminus Z_{alg}$ , the curvature of  $A(t)$  is uniformly bounded in  $C^0$  on  $K$ .*

Our results provide a partial generalization of the result of Daskalopoulos and Wentworth [3]. The main result of this paper is an equality of *sets*, while Daskalopoulos and Wentworth prove an equality of *analytic varieties*. The main difference here is a natural notion of algebraic multiplicity in the case that  $\dim_{\mathbb{C}} X = 2$ , which is attached to the set  $Z_{alg}$ . The precise result of [3] is that the mass of a bubble at a point  $p \in X$  is precisely equal to the algebraic multiplicity at  $p$ . The connection between these two quantities is provided by the Riemann-Roch theorem. However, the singularities of a torsion free coherent sheaf over a Kähler manifold  $X$  with  $\dim_{\mathbb{C}} X > 2$  are no longer isolated, and this poses a significant difficulty in generalizing the argument of [3]. As a result, the techniques we use are completely different. Before discussing the method of proof, we point out that Theorem 1 implies a new proof of the Donaldson-Uhlenbeck-Yau theorem, assuming Simpson's lower bound for the Donaldson functional [17].

**Corollary 1.** *Let  $E$  be an indecomposable vector bundle over a compact Kähler manifold, then  $E$  admits a Hermitian-Einstein connection if and only if  $E$  is stable in the sense of Mumford-Takemoto.*

In fact, we obtain something slightly more general,

**Corollary 2.** *Let  $(E, \bar{\partial}_0)$  be an indecomposable, holomorphic vector bundle over a compact Kähler manifold, and suppose that  $E$  admits a filtration of vector bundles*

$$0 = S^0 \subset S^1 \subset \cdots \subset S^p = E,$$

*with  $Q^i = S^i/S^{i-1}$  stable vector bundles. Then the Yang-Mills flow converges smoothly to a Yang-Mills connection on  $E_{\infty} = \bigoplus_i Q^i$ . In particular, the underlying smooth vector bundle  $E$  admits a Yang-Mills connection with a different holomorphic structure  $\bar{\partial} \neq \bar{\partial}_0$ .*

The interesting aspect of this corollary is that the Yang-Mills flow converges, despite the fact that the complex structure may jump in the limit. The phenomenon of jumping complex structure has proven to be an important and interesting difficulty in obtaining the convergence of the Kähler-Ricci flow; see for instance, the work of Phong-Sturm, and Phong-Song-Sturm-Weinkove [14, 15].

We now describe the idea behind the proof. For simplicity, assume that  $E$  is unstable and admits a coherent, torsion-free sheaf  $S$  with quotient  $Q$  so that both  $S$  and  $Q$  are stable, and  $\mu(S) > \mu(E) > \mu(Q)$ . Away from  $Z_{alg}$  we know  $S$  and  $Q$  are vector bundles, and so a choice of a metric  $H_0$  on  $E$  induces metrics  $J_0$  on  $S$  and  $M_0$  on  $Q$ . If  $\gamma$  is the second fundamental form associated to  $S$  and  $Q$  given by  $H_0$ , and  $F^S$  and  $F^Q$  are the curvatures of  $J_0$  and  $M_0$ , then we have the well known formula for the decomposition of the full curvature of  $E$ :

$$F = \begin{pmatrix} F^S - \gamma \wedge \gamma^\dagger & \tilde{\nabla}\gamma \\ -\tilde{\nabla}\gamma^\dagger & F^Q - \gamma^\dagger \wedge \gamma \end{pmatrix},$$

where  $\tilde{\nabla}$  denotes the induced connection on  $Hom(Q, S)$  (or  $Hom(S, Q)$  for  $\gamma^\dagger$ ). Since  $S$  and  $Q$  are stable, we expect that the induced geometry should be controlled uniformly. Indeed, this is the case. Using the stability of  $S$  and  $Q$  we are able to show that the quantities  $|F^Q|, |F^S|, |\gamma|$  are uniformly bounded on compact subsets  $K \Subset X \setminus Z_{alg}$ . At this point, it remains only to control  $|\tilde{\nabla}\gamma|$  uniformly on  $K$ . To achieve this, we use the relationship between the Donaldson heat flow and the Yang-Mills flow and a maximum principle argument. This requires some hard work since the quantities  $F^Q, F^S, \gamma$  which appear in the evolution equation for  $\tilde{\nabla}\gamma$  are only defined on  $X \setminus Z_{alg}$ . Applying the maximum principle on  $K$  yields a bound for  $|\tilde{\nabla}\gamma|$  only in terms of the values of  $|\tilde{\nabla}\gamma|$  on  $\partial K$ . This is clearly not sufficient. In order to deal with this difficulty we construct an explicit barrier function  $\sigma$  using the local algebra of the Harder-Narasimhan-Seshadri filtration, and find explicit estimates for the blow-up rate of the geometric quantities  $F^Q, F^S, \gamma$  near  $Z_{alg}$ . With these explicit estimates we are able to apply the maximum principle to a suitably scaled quantity involving the barrier function to obtain uniform control of  $\tilde{\nabla}\gamma$  on  $K \Subset X \setminus Z_{alg}$ . The general case is a simple modification of this argument. From this estimate Theorem 2, and hence Theorem 1, follow.

The organization of the paper is as follows. In section 2 we recall the basic objects we will need in the proof. In section 3 we use stability to obtain uniform bounds on the curvatures  $F^S, F^Q$  and  $\gamma$ . In section 4 we construct the barrier function  $\sigma$  and in section 5 we prove estimates on the blow-up rate of  $F^S, F^Q$  near  $Z_{alg}$ . In section 6 we compute the evolution equations of  $|\gamma|^2$  and  $|\tilde{\nabla}\gamma|^2$ , and in section 7 we apply the maximum principle to obtain uniform estimates for  $\tilde{\nabla}\gamma$  and complete the proof of Theorem 2.

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## 2 Background

### 2.1 The Yang-Mills flow and the Donaldson heat flow

We begin with a brief introduction to the Yang-Mills flow, and highlight the importance of its relation to the Donaldson heat flow.

Let  $X$  be a compact Kähler manifold, and assume the Kähler form  $\omega$  is normalized to have volume one. The Yang-Mills flow is a flow of connections  $d_A := d + A$  on  $E$ , where  $d_A : E \rightarrow E \otimes \Omega^1$ . Because  $X$  is a complex manifold, this map decomposes into  $(1, 0)$  and  $(0, 1)$  parts. In particular, the connection coefficients decompose as  $A = A' + A''$ , where  $A'$  represents the  $(1, 0)$  part and  $A''$  represents the  $(0, 1)$  part of  $A$ . Thus  $d_A = \partial_A + \bar{\partial}_A$ , where  $\partial_A := \partial + A'$  and  $\bar{\partial}_A := \bar{\partial} + A''$ . We say  $A$  is *integrable* if  $\bar{\partial}_A^2 = 0$ , which implies  $\bar{\partial}_A$  defines a holomorphic structure on  $E$ . For a fixed metric  $H_0$ , we say a connection is unitary if it is compatible with the metric, and we denote the space of integrable unitary connections by  $\mathcal{A}^{1,1}$ . The curvature of a connection, denoted  $F_A$ , is a section of  $\text{End}(E) \otimes \Omega^{1,1}$ , and is defined by:

$$F_A := \bar{\partial}A' + \partial A'' + A'' \wedge A' + A' \wedge A''.$$

The Yang-Mills functional  $YM : \mathcal{A}^{1,1} \rightarrow \mathbf{R}$  is defined to be the  $L^2$  norm of the curvature:

$$YM(A) := \|F_A\|_{L^2}^2.$$

On a general complex manifold, the Yang-Mills flow is the gradient flow of this functional, and is given by:

$$\dot{A} = -d_A^* F_A.$$

On a Kähler manifold we can rewrite the equation for the flow using Bianchi's second identity ( $d_A F_A = 0$ ) and the Kähler identities:

$$\begin{aligned} d_A^* F_A &= \partial_A^* F_A + \bar{\partial}_A^* F_A \\ &= i[\Lambda, \bar{\partial}_A] F_A - i[\Lambda, \partial_A] F_A \\ &= -i\bar{\partial}_A \Lambda F_A + i\partial_A \Lambda F_A. \end{aligned}$$

This gives:

**Definition 1.** *On a Kähler manifold, the Yang-Mills flow takes the form:*

$$\dot{A} = i\bar{\partial}_A \Lambda F_A - i\partial_A \Lambda F_A. \tag{2.2}$$

From this formulation one can check that if  $A(0) \in \mathcal{A}^{1,1}$ , then  $A(t)$  is an integrable, unitary connection for all time  $t \in [0, \infty)$ . Now if  $E$  is stable, it was first shown by Donaldson in [4] that the Yang-Mills flow converges to a Hermitian-Einstein connection. However, since we are assuming  $E$  is not stable, we do not expect the flow to converge to a limiting Hermitian-Einstein connection. In fact, our main object of study is the set of points on the base manifold  $X$  where the curvature blows up along the flow.

**Definition 2.** Given a sequence of connections  $A(t_j)$  along the Yang-Mills flow, the analytic singular set of  $E$  (sometimes called the bubbling set) is defined by:

$$Z_{an} = \bigcap_{r>0} \{x \in X \mid \liminf_{j \rightarrow \infty} r^{4-2n} \int_{B_r(x)} |F_A(t_j)|^2 \phi^n \geq \varepsilon\} \quad (2.3)$$

for some fixed  $0 < \varepsilon \leq \varepsilon_0$ , where  $\varepsilon_0 \ll 1$  depends only on  $X$ .

For a precise definition of  $\varepsilon_0$ , we direct the reader to the proof of Proposition 6 in [8]. In the Kähler setting, the Yang-Mills flow is closely related to the Donaldson heat flow. In fact, it is through this relationship that many important properties of the Yang-Mills flow were first realized, such as long time existence and convergence.

Along a solution to the Yang-Mills flow we fixed the metric  $H_0$  on  $E$  and evolved the connection (and hence the holomorphic structure). Alternatively, one can fix the holomorphic structure and evolve the metric, a viewpoint introduced by Donaldson in [4]. Starting with a fixed initial metric  $H_0$  on  $E$ , any other metric  $H$  is related to  $H_0$  by an endomorphism  $h = H_0^{-1}H$ . Conversely, any positive definite Hermitian endomorphism  $h$  defines a metric  $H = H_0 h$ .

**Definition 3.** Let  $\mathbf{1}$  denote the identity map in  $\text{End}(E)$ . The Donaldson heat flow is a flow of endomorphisms  $h = h(t)$  given by:

$$h^{-1} \dot{h} = -(\Lambda F - \mu(E)\mathbf{1}),$$

with initial condition  $h(0) = \mathbf{1}$ . Here,  $F$  is the curvature of the unitary Chern connection of the metric  $H(t) = H_0 h(t)$ .

A unique smooth solution of the flow exists for all  $t \in [0, \infty)$ , and on any stable bundle this solution will converge to a smooth Hermitian-Einstein metric [4], [5], [17], [18].

We now use a solution  $h(t)$  to the Donaldson heat flow to construct a solution  $A(t)$  to the Yang-Mills flow. Working in a unitary frame with respect to  $H_0$ , let  $A_0$  be an initial connection in  $\mathcal{A}^{1,1}$ . Starting with the initial holomorphic structure  $\bar{\partial}_0 := \bar{\partial} + A_0''$ , we consider the one parameter family of holomorphic structures  $\bar{\partial}_t = \bar{\partial} + A_t''$ , where  $A_t''$  is defined by the action of  $w = h^{1/2}$  on  $A_0''$ . Explicitly, this action is given by:

$$A_t'' = w A_0'' w^{-1} - \bar{\partial} w w^{-1}, \quad (2.4)$$

which is equivalent to:

$$\bar{\partial}_t := w \circ \bar{\partial}_0 \circ w^{-1}.$$

Using this one-parameter family of holomorphic structures and the metric  $H_0$ , we define a one-parameter family of unitary connections  $A_t$ , and one can check that  $A_t$  evolves by the Yang-Mills flow. Conversely, any one-parameter path in  $\mathcal{A}^{1,1}$  along the Yang-Mills flow defines an orbit of the complexified gauge group, which gives rise to a solution of the

Donaldson heat flow. The curvature of  $F$  along the Donaldson heat flow is related to the curvature  $F_A$  along the Yang-Mills flow by the following relation:

$$F_A = w F w^{-1}. \quad (2.5)$$

An important consequence of this relationship is that the norm of the curvature along the Yang Mills flow given by the fixed metric  $H_0$  is equivalent to the norm of the curvature along the Donaldson heat flow given by the evolving metric  $H$ . Let  $(\cdot)^\dagger$  denote the adjoint of an endomorphism with respect to the fixed metric  $H_0$ , and let  $(\cdot)^*$  denote the adjoint with respect to the evolving metric  $H$ . For any endomorphism  $M$ , these two adjoints are related as follows:  $M^\dagger = h M^* h^{-1}$ . We then see:

$$|F|_H^2 = \text{Tr}(F F^*) = \text{Tr}(w^{-1} F_A w (w^{-1} F_A w)^*) = \text{Tr}(F_A h F_A^* h^{-1}) = \text{Tr}(F_A F_A^\dagger) = |F_A|_{H_0}^2.$$

Thus from the point of view of uniform curvature bounds, it suffices to prove bounds along either the Donaldson heat flow or the Yang-Mills flow, provided we always compute the norm with the right metric.

We conclude this section with a simple curvature bound along the Donaldson heat flow.

**Lemma 1.** *Along the Donaldson heat flow, there is a constant  $C$  so that  $|\Lambda F|_{H(t)}$  is uniformly bounded.*

*Proof.* We have the following simple computation for the heat operator on  $|\Lambda F|_H^2$  (for details see [9]):

$$(\partial_t - \Delta)|\Lambda F|_H^2 = -|\nabla \Lambda F|_H^2 - |\bar{\nabla} \Lambda F|_H^2 \leq 0.$$

The lemma follows from the maximum principle.  $\square$

## 2.2 Quotients, filtrations and stability

In this section we introduce the algebraic singular set, show it is uniquely determined by the isomorphism class of  $E$ , and provide a local analytic description. We begin by recalling the definitions of slope and stability.

Given a torsion free sheaf  $\mathcal{E}$ , we can define its first Chern class by  $c_1(\mathcal{E}) := c_1(\det(\mathcal{E}))$ , since  $\det(\mathcal{E})$  is always a line bundle. The slope of  $\mathcal{E}$  is then given by:

$$\mu(\mathcal{E}) := \frac{1}{rk(\mathcal{E})} \int_X c_1(\det(\mathcal{E})) \wedge \omega^{n-1}.$$

We say  $\mathcal{E}$  is stable if for every torsion free subsheaf  $\mathcal{F} \subset \mathcal{E}$  the inequality  $\mu(\mathcal{F}) < \mu(\mathcal{E})$  holds.  $\mathcal{E}$  is semi-stable if the weak inequality  $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$  holds.

Next we introduce the Harder-Narasimhan filtration, and recall some of its properties. The following proposition can be found in [12].

**Proposition 1** ([12], Theorem (7.15)). *Any torsion-free sheaf  $E$  carries a unique filtration of subsheaves*

$$0 = S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^p = E, \quad (2.6)$$

*called the Harder-Narasimhan filtration of  $E$ , such that the quotients  $Q^i = S^i/S^{i-1}$  are torsion-free and semi-stable. Moreover, the quotients are slope decreasing, satisfying  $\mu(Q^i) > \mu(Q^{i+1})$ , and the associated graded object  $Gr^{hn}(E) := \bigoplus_{i=1}^p Q^i$  is uniquely determined by the isomorphism class of  $E$ .*

We sometimes abbreviate this filtration as the HN filtration. For our purposes having semi-stable quotients is not good enough, and we must take the filtration one step further:

**Proposition 2** ([12], Theorem (7.18)). *Given a semi-stable sheaf  $\mathcal{Q}$ , there exists a filtration by subsheaves, called the Seshadri filtration:*

$$0 = \tilde{S}^0 \subset \tilde{S}^1 \subset \cdots \subset \tilde{S}^q = \mathcal{Q},$$

*such that  $\mu(\tilde{S}^i) = \mu(\mathcal{Q})$  for all  $i$ , and each quotient  $\tilde{Q}^i = \tilde{S}^i/\tilde{S}^{i-1}$  is torsion-free and stable. Furthermore, the direct sum of the stable quotients, denoted  $Gr^s(\mathcal{Q}) := \bigoplus_{i=1}^q \tilde{Q}^i$ , is canonical and uniquely determined by the isomorphism class of  $\mathcal{Q}$ .*

Combining these two propositions, we can construct the Harder-Narasimhan-Seshadri filtration, by finding a Seshadri filtration for each semi-stable quotient in the HN filtration. We sometimes refer to this double filtration as the HNS filtration. Consider the direct sum of stable quotients:

$$Gr^{hns}(E) := \bigoplus_k \bigoplus_i \tilde{Q}_k^i.$$

It is not hard to check that the Harder-Narasimhan-Seshadri filtration can be written as a single filtration of  $E$  by torsion-free coherent sheaves:

$$0 = S^0 \subset S^1 \subset \cdots \subset S^p = E, \quad (2.7)$$

and in this case, setting  $Q^i = S^i/S^{i-1}$  we have  $Gr^{hns}(E) = \bigoplus_{i=1}^{p-1} Q^i$ . We are now ready for the following definition:

**Definition 4.** *The algebraic singular set is defined to be:*

$$Z_{alg} := \{x \in X \mid Gr^{hns}(E)_x \text{ is not free}\}.$$

We would like to elucidate this definition by providing a useful local description of the algebraic singular set. For the moment, let us focus on the simple case when  $E$  has a stable subsheaf  $S$  with stable quotient  $Q$ , so the HNS filtration of  $E$  is given by:

$$0 \subset S \subset E.$$



In this case, we have the exact sequence of torsion free, coherent sheaves

$$0 \longrightarrow S \xrightarrow{B} E \xrightarrow{p} Q \longrightarrow 0, \quad (2.8)$$

and  $Gr^{hns}(E) = S \oplus Q$ . Since  $S$  is coherent, the inclusion  $S \hookrightarrow E$  is locally given by a matrix of holomorphic functions  $B = B^\alpha_\beta$ . Moreover, over a sufficiently small open set  $U \subset X$ ,  $S$  has a finite length resolution

$$0 \longrightarrow \mathcal{O}_U^{\oplus r_\ell} \longrightarrow \mathcal{O}_U^{\oplus r_{\ell-1}} \longrightarrow \dots \longrightarrow \mathcal{O}_U^{\oplus r_1} \xrightarrow{T} S \longrightarrow 0. \quad (2.9)$$

The surjection  $\mathcal{O}_U^{\oplus r_1} \rightarrow S$  is given by a matrix of holomorphic functions  $T = T^\gamma_\delta$ . The resolution of  $S$  gives rise to a resolution for  $Q$

$$0 \longrightarrow \mathcal{O}_U^{\oplus r_\ell} \longrightarrow \mathcal{O}_U^{\oplus r_{\ell-1}} \longrightarrow \dots \longrightarrow \mathcal{O}_U^{\oplus r_1} \xrightarrow{B \circ T} E \longrightarrow Q \longrightarrow 0,$$

where now the map  $\mathcal{O}_U^{\oplus r_1} \rightarrow E$  is the composition  $B \circ T$ , which is determined locally by the matrix product  $B \circ T = B^\alpha_\gamma T^\gamma_\delta$ . The main technical result we need is the following theorem:

**Theorem 3** ([12] Chapter 5, Theorem 5.8). *Let  $\zeta$  be a coherent sheaf,  $U \subset X$  an open set over which  $\zeta$  has a finite resolution*

$$0 \longrightarrow \mathcal{O}_U^{\oplus r_\ell} \longrightarrow \dots \longrightarrow \mathcal{O}_U^{\oplus r_2} \xrightarrow{h} \mathcal{O}_U^{\oplus r_1} \longrightarrow \zeta_U \longrightarrow 0.$$

*Then we have the following equality of sets;*

$$\{x \in U \mid \zeta_x \text{ is not free}\} = \{x \in U \mid \text{rank}(h(x)) < \max_{y \in U} \text{rank}(h(y))\}. \quad (2.10)$$

In particular, it follows immediately that  $Z_{alg}$  is an analytic subset of  $X$ . In our setting, we note that any point where  $Q_x$  is free, the stalk  $S_x$  is free as well. In particular, we have,

**Corollary 3.** *On a sufficiently small neighborhood,  $Z_{alg}$  is given by:*

$$Z_{alg} \cap U = \{x \in U \mid \text{rank}(B \circ T(x)) < \max_{y \in U} \text{rank}(B \circ T)(y)\}$$

In the general case we obtain a similar description inductively. Recall the filtration (2.7). For each  $i \leq p$  we let  $B_i$  be inclusion map

$$0 \longrightarrow S^{i-1} \xrightarrow{B_i} S^i \longrightarrow Q^i \longrightarrow 0, \quad (2.11)$$

Denote  $Z_i$  the set where  $Q^i$  fails to be locally free. The set  $Z_p$  was described in the simple case above. Fix an open set  $U \subset X \setminus Z_p$ . On  $U$ ,  $S^{p-1}$  is a vector bundle, and hence we get a resolution of  $Q^{p-1}$

$$0 \longrightarrow \mathcal{O}_U^{\oplus r_\ell} \longrightarrow \dots \longrightarrow \mathcal{O}_U^{\oplus r_1} \xrightarrow{B_{p-1} \circ T_{p-1}} S^{p-1} \longrightarrow Q^{p-1} \longrightarrow 0.$$

We thus obtain a description of the set  $\tilde{Z}_{p-1} = Z_{p-1} \cap X \setminus Z_p$ . Then  $Z_{p-1} = \tilde{Z}_{p-1} \cup Z_p$  is precisely the set where  $Q^p \oplus Q^{p-1}$  fails to be locally free. This continues inductively. As an example, we will indicate how to obtain a description of  $Z_{alg}$  in the case where the  $HNS$  filtration of  $E$  has three quotients,  $0 \subset S^1 \subset S^2 \subset E$ . In this case, we have  $Gr^{hns}(E) = S^1 \oplus S^2/S^1 \oplus E/S^2$ . Using the above argument we obtain an explicit local description of the set

$$Z_2 = \{x \in X \mid E/S^2_{(x)} \text{ is not free}\}.$$

Now, consider the exact sequence:

$$0 \longrightarrow S^1 \longrightarrow S^2 \longrightarrow S^2/S^1 \longrightarrow 0.$$

The main difference between this sequence and the sequence (2.8) is that  $S^2$  is not a vector bundle. However, it is a vector bundle over  $X \setminus Z_2$ . Thus, working over this open manifold we can find an explicit local description of  $Z_1 = \{x \in X \mid S^2/S^1_{(x)} \text{ is not free}\}$ . Since we clearly have  $Z_{alg} = Z_1 \cup Z_2$ , we have succeeded in obtaining a local description of  $Z_{alg}$  in this case.

### 2.3 The induced geometry of subsheaves and quotient sheaves

As discussed in the introduction, we prove the main theorem by decomposing the Yang-Mills flow on  $E$  into flows on stable subsheaves and stable quotient sheaves. In this section we define induced metrics and provide explicit formulas for the induced connections we will need later on. We recall the exact sequence (2.8) and restrict ourselves to the open manifold  $X \setminus Z_{alg}$ . Because the sheaves  $S$  and  $Q$  are locally free here, the metric  $H_0$  on  $E$  induces a metric  $J$  on  $S$  and a metric  $M$  on  $Q$ . For sections  $\psi, \phi$  of  $S$ , we define the metric  $J$  as follows:

$$\langle \phi, \psi \rangle_J = \langle B(\phi), B(\psi) \rangle_{H_0}.$$

In order to define  $M$  on  $Q$ , we note that  $H_0$  gives a splitting of (2.8):

$$0 \longleftarrow S \xleftarrow{\pi} E \xleftarrow{p^\dagger} Q \longleftarrow 0. \quad (2.12)$$

Here  $\pi$  is the orthogonal projection from  $E$  onto  $S$  with respect to the metric  $H_0$ . For sections  $v, w$  of  $Q$ , we define the metric  $M$  by:

$$\langle v, w \rangle_M = \langle p^\dagger(v), p^\dagger(w) \rangle_{H_0}.$$

**Definition 5.** On  $X \setminus Z_{an}$  the sheaves  $S^i$  and  $Q^i$  are holomorphic vector bundles. We define an *the induced metric*  $J_i$  on  $S^i$ , and  $K_i$  on  $Q^i$  to be one constructed as above.

Note that on  $X \setminus Z_{alg}$  it is equivalent to induce the metric  $J_{i-1}$  on  $S^{i-1}$  by restricting the metric  $J_i$  induced on  $S^i$  to the image of  $S^{i-1} \subset S^i$ .

Once we have sequence (2.12), the second fundamental form  $\gamma \in \Gamma(X, \Lambda^{0,1} \otimes Hom(Q, S))$  is given by:

$$\gamma = \bar{\partial} p^\dagger.$$

Of course, by composing with the projection  $p$ , we can write the second fundamental form as a homomorphism from  $S^\perp$  to  $S$ :  $\gamma \circ p = \bar{\partial} p^\dagger \circ p$ . As  $p$  is holomorphic, and  $p^\dagger \circ p = \mathbb{1} - \pi$ , we see  $\gamma \circ p = \bar{\partial}(\mathbb{1} - \pi) = -\bar{\partial}\pi$ . By the definition of the induced metric  $M$ , working with  $\gamma$  and  $\gamma \circ p$  are equivalent once we take corresponding norms, so we suppress the map  $p$  from our notation.

We now establish some formulas for the induced connections on  $S^i$  and  $Q^i$ . In order to avoid confusion we shall temporarily implement the practice of using Roman indices for  $S^{i-1}$  and Greek indices for  $S^i$ . The connection on  $S^{i-1}$  is induced from the connection on  $S^i$  by the inclusion  $B_i : S^{i-1} \rightarrow S^i$  and the projection  $\pi^i : S^i \rightarrow S^{i-1}$  obtained from the metric  $J_i$ . In order to find a local formula for the connection coefficients on  $S^{i-1}$ , we first note that  $\pi^i : S^i \rightarrow S^{i-1}$  on  $U \subset X \setminus Z_{alg}$  is determined by the formula

$$(\pi^i)^k{}_\beta = (J_i)_{\bar{\alpha}\beta} \overline{(B_i)^{\alpha}{}_\ell} (J_{i-1})^{\bar{\ell}k}. \quad (2.13)$$

From this formula, we obtain the following lemma by a straightforward computation:

**Lemma 2.** *The anti-holomorphic component of the induced connection on  $S^{i-1}$  over  $X \setminus Z_{alg}$  is given by*

$$(A''^{S^{i-1}})^p{}_\ell = (J_i)_{\bar{\alpha}\gamma} \overline{(B_i)^{\alpha}{}_m} (J_{i-1})^{\bar{m}p} (A''^{S^i})^\gamma{}_\beta (B_i)^\beta{}_\ell.$$

The  $(1,0)$  component of the connection is obtained by unitarity with respect to the induced metric  $J_{i-1}$ . The curvature  $F^{S^{i-1}}$  is then computed using this induced connection.

We can similarly compute on the bundle  $Q^i$ . Observe that local identification  $Q^i \cong \text{im}(S^{i-1})^\perp$  defined by  $J_i$  allows us to compute on  $(S^{i-1})^\perp$  inside  $S^i$ , since the metric on  $Q^i$  is obtained precisely via this identification. Using equation (2.13), we compute

$$\begin{aligned} (A''^{Q^i})^\nu{}_\alpha &= (\mathbb{1} - \pi)^\nu{}_\delta (A''^{S^i})^\delta{}_\alpha \\ &= (A''^{S^i})^\nu{}_\alpha - (J_i)_{\bar{\gamma}\beta} \overline{(B_i)^{\gamma}{}_\ell} (B_i)^\nu{}_k (J_{i-1})^{\bar{\ell}k} (A''^{S^i})^\beta{}_\alpha, \end{aligned} \quad (2.14)$$

where as before  $\mathbb{1} \in \text{End}(S^i)$  denotes the identity map. These formulae allow us to inductively determine the connection coefficients on  $S^i$  by using that  $S^p = E$  and  $J_p = H_0$ .

### 3 Bounds from stability

In this section we use the stability of the quotients of the Harder-Narasimhan-Seshadri filtration to get bounds on the induced curvature. For simplicity we first consider the case where the HNS filtration is a single step  $0 \subset S \subset E$ . Here  $S$  has quotient  $Q$  and  $Gr^{hns}(E) = S \oplus Q$ . At the end of this section we discuss the general case.

Consider a family of connections  $A(t)$  evolving along the Yang-Mills flow, with corresponding evolving holomorphic structure  $\bar{\partial}_t := w(t) \circ \bar{\partial}_0 \circ w(t)^{-1}$ . Let  $F_A(t)$  be the curvature of  $A(t)$ . The metric  $H_0$  induces metrics  $J_0$  on  $S$  and  $M_0$  on  $Q$ , and the holomorphic structure  $\bar{\partial}_t$  restricts to holomorphic structures on  $S$  and  $Q$ , inducing curvatures  $F_A(t)^S$  and  $F_A(t)^Q$ .

**Proposition 3.** *Let  $A(t)$  be a family of connections evolving along the Yang-Mills flow. Then, for any compact subset  $K \subset X \setminus Z_{alg}$  the induced curvatures  $F_A(t)^S$  and  $F_A(t)^Q$  are uniformly bounded in  $C^k(K)$  for any  $k$ .*

In section 5 we will obtain precise estimates for the rate of blow-up near  $Z_{alg}$ . As a first step, we relate the projections evolving along the Donaldson heat flow to projections evolving along the Yang-Mills flow. In the case of the Donaldson heat flow, consider the orthogonal projection  $\pi_t$  onto a fixed subsheaf  $S \subset E$ , which evolves due to the fact that the metric  $H$  is changing. Along the Yang-Mills flow the metric  $H_0$  is fixed, yet the subsheaf  $S$  is acted on by the complexified gauge transformation  $w$ . Thus the orthogonal projection  $\pi_w$  onto  $w(S)$  evolves as well. In order to lighten notation, we shall denote  $w(t)$  by  $w$  when no confusion will result.

**Lemma 3.** *The two evolving projections are related as follows:*

$$\pi_w = w\pi_t w^{-1}$$

*Proof.* It is immediately clear that  $(w\pi_t w^{-1})^2 = w\pi_t w^{-1}$ , so  $w\pi_t w^{-1}$  is a projection onto the subsheaf  $w(S)$ . We complete the lemma by showing it is unitary with respect to  $H_0$ .

$$(w\pi_t w^{-1})^\dagger = w^{-1}(\pi_t)^\dagger w = h^{-1/2} h \pi_t^* h^{-1} h^{1/2} = w\pi_t w^{-1}.$$

□

Next we show the complex gauge transformations  $w(t_j)$  restrict to gauge transformations of  $S$  and  $Q$  with no diagonal terms.

**Lemma 4.** *The gauge transformations  $w(t)$  decomposes as follows onto  $S$  and  $Q$ :*

$$w(t) = \begin{pmatrix} w^S(t) & 0 \\ 0 & w^Q(t) \end{pmatrix}.$$

*Proof.* Let  $\phi$  be a section of  $S$ . Observe that  $w(\phi)$  remains a section of  $w(S)$  since

$$\pi_w(w(\phi)) = w\pi_t w^{-1}w(\phi) = w\pi_t(\phi) = w(\phi).$$

Thus no component of  $w$  maps  $S$  to its perpendicular space. The lemma follows since  $w$  is Hermitian, so the off diagonal terms are equal. □

Consider the standard decomposition of the holomorphic structure:

$$\bar{\partial}_0 = \begin{pmatrix} \bar{\partial}_0^S & \gamma_0 \\ 0 & \bar{\partial}_0^Q \end{pmatrix},$$

where  $\gamma_0$  is the second fundamental form associated to  $S$  defined by  $H_0$ . From now on, we work along a subsequence  $t_j$ . The above lemma shows the evolving holomorphic structures on  $S$  and  $Q$  are given by:

$$\bar{\partial}_j^S = w^S(t_j) \circ \bar{\partial}_0^S \circ (w^S(t_j))^{-1} \quad \text{and} \quad \bar{\partial}_j^Q = w^Q(t_j) \circ \bar{\partial}_0^Q \circ (w^Q(t_j))^{-1}.$$

Consider the normalized endomorphisms:

$$\tilde{w}^S(t_j) := \frac{w^S(t_j)}{\|w^S(t_j)\|_{L^2(X)}} \quad \text{and} \quad \tilde{w}^Q(t_j) := \frac{w^Q(t_j)}{\|w^Q(t_j)\|_{L^2(X)}},$$

Note that the actions of  $\tilde{w}^S$  and  $w^S$  produce the same complex structures (similarly for  $\tilde{w}^Q$  and  $w^Q$ ). Our goal is to show that these normalized endomorphisms do not degenerate as  $j$  tends to infinity, and that they converge (along a subsequence  $t_{j_k}$  and away from  $Z_{alg}$ ) to limiting endomorphisms  $w_\infty^S$  and  $w_\infty^Q$  in  $C^k$ . This gives uniform  $C^k(X)$  bounds on both  $\tilde{w}^S(t_{j_k})$  and  $(\tilde{w}^S)^{-1}(t_{j_k})$  (similarly for  $\tilde{w}^Q(t_{j_k})$  and  $(\tilde{w}^Q)^{-1}(t_{j_k})$ ), and thus bounds on the holomorphic structures  $\bar{\partial}_{j_k}^S$  and  $\bar{\partial}_{j_k}^Q$ . From these estimates Proposition 3 follows easily.

From now on we work exclusively with  $S$ , as the bounds for  $Q$  follow in a similar fashion. Also, we denote  $\tilde{w}^S$  by  $\tilde{w}$ , for notational simplicity. Finally, all the norms in the following proposition are computed with the fixed metric  $H_0$ .

**Proposition 4.** *Let  $t_j$  be a sequence of times along the Yang-Mills flow. Then there exists a subsequence  $t_{j_k}$  where the complex gauge transformations  $\tilde{w}(t_{j_k})$  are uniformly bounded in  $C^k(X \setminus Z_{alg})$ . Furthermore, after passing to a further subsequence, they converge in  $C^0$  to a limiting holomorphic map  $w_\infty$ , which is non-trivial.*

By “uniformly bounded in  $C^k(X \setminus Z_{alg})$ ”, we mean that we have bounds on every compact set  $K \Subset X \setminus Z_{alg}$ , which do not depend on  $K$ .

*Proof.* The idea of the proof is very similar to an argument used in [10], which has roots in an argument from [4]. We include the details here for the readers convenience. Also, for notational simplicity all subsequences will still be denoted  $t_j$ .

Consider the family of compact subsets  $K(r) \subset X \setminus Z_{an}$ , defined to be the complement of a tube around  $Z_{an}$ :

$$K(r) := X \setminus \bigcup_{x \in Z_{an}} B_r(x).$$

First, we show that for some  $r_0 > 0$ ,

$$\|\tilde{w}(t_j)\|_{C^0(X \setminus Z_{alg})} \leq C \|\tilde{w}(t_j)\|_{L^2(K(r_0))}, \quad (3.15)$$

where the constant  $C$  is uniform in  $j$ . Let  $A(t_j)$  be the initial sequence of connections on  $S$ . By the convergence results of Hong-Tian [8], there exists a subsequence  $A''(t_j)$  which converges smoothly to a limiting holomorphic structure  $A''_\infty$  in  $K(r)$ . By the definition of  $\bar{\partial}_j^S$ ,  $\tilde{w}(t_j)$  solves the equation:

$$\bar{\partial}^\dagger (\bar{\partial} \tilde{w}(t_j) + A''(t_j) \tilde{w}(t_j) - \tilde{w}(t_j) A''_0) = 0, \quad (3.16)$$

which is an elliptic equation in divergence form. After multiplying  $\tilde{w}_j$  by a suitable bump function which vanishes along  $\partial K(r)$  and is identically one in  $K(2r)$ , and applying the uniform bounds on  $A''(t_j)$ , we can use the Moser iteration technique to get:

$$\|\tilde{w}\|_{C^0(K(2r))} \leq C(r) \|\tilde{w}\|_{L^2(K(r))}, \quad (3.17)$$

(for details see [7] Theorem 8.15). However,  $C(r) \rightarrow \infty$  as  $r$  approaches zero. Nevertheless, for a fixed  $r$  we have uniform control of  $\tilde{w}$  in the set  $K(2r)$  where the bump function is identically one. We claim this bound controls the  $C^0$  norm of  $\tilde{w}$  on all of  $X \setminus Z_{alg}$ . Consider the most general assumption on the singular set; that  $Z_{an}$  is closed and has finite  $(2n-4)$ -Hausdorff measure, denoted  $H^{2n-4}(Z_{an}) \leq C$  (see Proposition 6 from [8]). It follows that  $H^{2(n-1)}(Z_{an}) = 0$ , and by a result of Shiffman (Lemma 2 from [16]), for every point  $x \in X \setminus Z_{an}$ , almost all complex lines through  $x$  do not intersect  $Z_{an}$ . Thus working in local coordinates, we construct a lattice  $L$  with edges formed by complex lines such that  $L \cap Z_{an} = \emptyset$ . Since both  $L$  and  $Z_{an}$  are compact, they are separated by a finite distance, so in particular there exists an  $r_0$  such that  $L \cap X \setminus K(r_0) = \emptyset$ . By (3.17) the maps  $\tilde{w}$  are uniformly bounded in  $C^0$  along the complex lines that form  $L$ . Since the  $\tilde{w}$  are holomorphic maps and defined on all of  $X \setminus Z_{alg}$  (this is where  $S$  is a vector bundle), locally they are given by a matrix of holomorphic functions. Since  $Z_{alg}$  is of at least complex codimension 2, we know none of the holomorphic functions blow up along that subvariety, thus we just consider  $\tilde{w}$  as a function on all of  $X$ , and note that in the end of the argument we get bounds where  $\tilde{w}$  was originally defined. By applying Cauchy's integral formula along each line to each entry of the matrix we extend this uniform bound to each face of  $L$ , and after repeated applications of Cauchy's integral formula, the  $C^0$  bound extends to the interior of  $L$ . Since  $X$  is compact it can be covered by the interiors of finitely many lattices, thus there exists a single  $r_0$  that works for all of  $X$ . The  $C^0$  bound on all of  $X$  follows, proving (3.15). Moreover, for any  $\rho < r_0$ , we have

$$\|\tilde{w}(t_j)\|_{C^0(X)} \leq C \|\tilde{w}(t_j)\|_{L^2(K(r_0))} \leq C \|\tilde{w}(t_j)\|_{L^2(K(\rho))}, \quad (3.18)$$

and the constant  $C$  is independent of  $\rho$ .

Now that we have a  $C^0$  bound, by the Cauchy estimates we get the  $C^k$  bounds on  $X \setminus Z_{alg}$ . By Arzelà-Ascoli, the sequence  $\tilde{w}(t_j)$  converges along a subsequence to a limiting map  $w_\infty$ . We claim that  $w_\infty$  is in fact holomorphic. Let  $\bar{\partial}_{j,0}$  denote the holomorphic structure on endomorphisms of  $S$  intertwining  $A''(t_j)$  and  $A''(0)$ . As we have seen  $\bar{\partial}_{j,0}\tilde{w}(t_j) = 0$ , so

$$\bar{\partial}_{\infty,0}w_\infty = \bar{\partial}_{j,0}(w_\infty - \tilde{w}(t_j)) - (\bar{\partial}_{j,0} - \bar{\partial}_{\infty,0})w_\infty,$$

and hence

$$\|\bar{\partial}_{\infty,0}w_\infty\|_{L^p(K(\rho))} \leq \|w_\infty - \tilde{w}(t_j)\|_{L_1^p(K(\rho))} + \|A(t_j) - A_\infty\|_{L^q(K(\rho))}^\lambda \|w_\infty\|_{L^r(K(\rho))}^{1-\lambda},$$

where  $q$ ,  $r$ , and  $\lambda$  are given by Holder's inequality. The left hand side is independent of  $j$ , so sending  $j$  to infinity we obtain

$$\|\bar{\partial}_{\infty,0}w_\infty\|_{L^p(K(\rho))} = 0$$

for any  $p$ . By elliptic regularity  $w_\infty$  is smooth, and thus holomorphic.

In fact,  $w_\infty$  is holomorphic on all of  $X \setminus Z_{an}$ . Pick any point  $x_0 \in X \setminus Z_{an}$ . Then there exists a  $\rho' < \rho$  such that  $K(\rho')$  contains  $x_0$ . By choosing the sequence  $\tilde{w}_j$  from above,

and repeating the convergence argument for the compact set  $K(\rho')$  as opposed to  $K(\rho)$ , we get convergence along a subsequence to a new holomorphic map  $w'_\infty$  defined on all of  $K(\rho')$ . Since we chose a subsequence of our original sequence, it follows that  $w'_\infty = w_\infty$  on  $K(\rho)$ , thus  $w_\infty$  extends holomorphically to all of  $K(\rho')$ . Repeating this argument for each point in  $X \setminus Z_{an}$ , we conclude that  $w_\infty$  is holomorphic everywhere in  $X \setminus Z_{an}$ . Since  $H^{2n-2}(Z_{an}) = 0$ , by Lemma 3 in [16],  $w_\infty$  extends to a holomorphic map on all of  $X$ .

Next we show  $w_\infty$  is not identically zero on  $K(\rho)$ . From the  $C^0$  estimate (3.18) one can show  $\|\tilde{w}(t_j)\|_{L^2(X \setminus K(\rho))}^2 \leq C \cdot \text{Vol}(X \setminus K(\rho))$ , where  $C$  is independent of  $j$  and the choice of  $\rho$ . Choose  $\rho$  small so that  $\text{Vol}(X \setminus K(\rho)) < 1/(2C)$ . It follows that

$$\|\tilde{w}(t_j)\|_{L^2(K(\rho))}^2 = \|\tilde{w}(t_j)\|_{L^2(X)}^2 - \|\tilde{w}(t_j)\|_{L^2(X \setminus K(\rho))}^2 \geq \frac{1}{2},$$

using our normalization  $\|\tilde{w}(t_j)\|_{L^2(X)}^2 = 1$ . This completes the proof of Proposition 4.  $\square$

We claim that  $w_\infty$  has no kernel. Note we are still working on  $S$ , and  $w_\infty$  is the  $C^0$  limit of the normalized endomorphism  $\tilde{w}^S(t_j)$ . So  $w_\infty$  is a holomorphic map of sheaves from  $S$  to  $\hat{E}_\infty$  (here  $\hat{E}_\infty$  is the reflexive extension of  $E_\infty$  to all of  $X$ ). First we show that the image of  $w_\infty$  is a subsheaf of  $E_\infty$  of the same degree and rank as  $S$ . By Theorem 1 from [11], combined with inequality (4.18) in that same reference, we know the  $L^2$  norm of the second fundamental form associated to  $S$  goes to zero. By Proposition 10 from [11], the projection  $\pi_j$  onto  $w(t_j)(S)$  converges to a limiting holomorphic splitting  $\pi_\infty$  of  $E_\infty$  into the direct sum  $S_\infty \oplus Q_\infty$ , where  $S_\infty$  has the same rank as  $S$ . This convergence is smooth away from  $Z_{an}$ . Furthermore, since  $\pi_j \circ \tilde{w}^S(t_j) = \tilde{w}^S(t_j)$  along the flow,  $\pi_\infty w_\infty = w_\infty$  and thus:

$$w_\infty : S \longrightarrow S_\infty.$$

Now,  $\Lambda F$  converges to  $\Lambda F_\infty$  away from  $Z_{an}$ , and by the proof of Lemma 4 in [11] it follows that  $\Lambda F_\infty|_{S_\infty} = \mu(S)\mathbb{1}$ . By Theorem 2 in [1] we know  $\Lambda F_\infty|_{S_\infty}$  extends to where  $S_\infty$  is locally free and by Theorem 3 in [1] the sheaf  $S_\infty$  is semi-stable. So  $w_\infty$  is a holomorphic map from a stable sheaf to a semi-stable sheaf of the same degree and rank, and thus must be an isomorphism. It follows that the rank of  $w_\infty$  does not drop.

Since  $\tilde{w}^S(t_j)$  converges to  $w_\infty$  in  $C^0$  and the rank of  $w_\infty$  does not drop, the lowest eigenvalue of  $\tilde{w}^S(t_j)$  is bounded from below. We can follow the same argument for the normalized endomorphisms  $\tilde{w}^Q$  on  $Q$ . Thus we have shown the following important proposition:

**Proposition 5.** *Given any sequence of times  $t_j$ , there exists a subsequence  $t_{j_k}$  and a uniform constant  $C_q$  such that, for any compact sect  $K \Subset X \setminus Z_{alg}$  the following estimates hold:*

$$\begin{aligned} \|\tilde{w}^S(t_{j_k})\|_{C^q(K)} &\leq C_q & \|(\tilde{w}^S)^{-1}(t_{j_k})\|_{C^q(K)} &\leq C_q, \\ \|\tilde{w}^Q(t_{j_k})\|_{C^q(K)} &\leq C_q & \|(\tilde{w}^Q)^{-1}(t_{j_k})\|_{C^q(K)} &\leq C_q. \end{aligned}$$

The crucial point here is that  $C_q$  *does not* depend on  $K$ . We would like to point out that the main ingredient required for the above argument is the lower boundedness of the Donaldson functional, due to Simpson [17], and extended by the second author in [9, 11]. Alternatively, one could argue as in [28], using multiplier ideal sheaves, however this argument also requires lower bounds on the Donaldson functional. Since the above estimate holds for *any* subsequence, we obtain

**Proposition 6.** *For any positive integer  $q$  there is a constant  $C_q$  such that, for any  $K \Subset X \setminus Z_{alg}$  the following estimates hold uniformly along the Yang-Mills flow*

$$\begin{aligned} \|\tilde{w}^S(t)\|_{C^q(K)} &\leq C_q & \|(\tilde{w}^S)^{-1}(t)\|_{C^q(K)} &\leq C_q, \\ \|\tilde{w}^Q(t)\|_{C^q(K)} &\leq C_q & \|(\tilde{w}^Q)^{-1}(t)\|_{C^q(K)} &\leq C_q. \end{aligned}$$

*Proof.* We argue by contradiction. Suppose the conclusion of the proposition is false. Then, there exists a subsequence of times  $t_j$  along the Yang-Mills flow where, for example,

$$\|\tilde{w}^S(t_j)\|_{C^q(K)} \geq j. \quad (3.19)$$

We now apply the previous proposition to find a subsequence  $t_{j_k}$  and a constant  $C_q$  such that

$$\|\tilde{w}^S(t_{j_k})\|_{C^q(K)} < C_q, \quad (3.20)$$

which clearly provides a contradiction.  $\square$

We can now prove Proposition 3. Let  $K$  be a compact set away from  $Z_{alg}$ . Then on  $K$  the induced holomorphic structures  $\bar{\partial}_0^S$  and  $\bar{\partial}_0^Q$  and induced metrics on  $S$  and  $Q$  are smooth. By the previous proposition and the definition of the evolving induced connections we see that  $A^S(t)$  and  $A^Q(t)$  are bounded in  $C^k(K)$ , which proves Proposition 3. As a corollary we get a  $C^0$  bound for the second fundamental form.

**Corollary 4.** *Let  $A(t)$  be a family of connections evolving along the Yang-Mills flow. Then on any compact subset  $K$  away from  $Z_{alg}$  the second fundamental form  $\gamma$  is bounded uniformly in  $C^0(K)$ .*

*Proof.* Fix  $K \Subset X \setminus Z_{alg}$ . The norm of  $\gamma$  is given by:

$$|\gamma|^2 = g^{j\bar{k}} \text{Tr}(\gamma_j^\dagger \gamma_{\bar{k}}),$$

where the adjoint  $\gamma^\dagger$  is computed with respect to the fixed metric  $H_0$ . By the standard decomposition of curvature  $F$  onto  $S$  we have

$$F|_S = F^S - \gamma \wedge \gamma^\dagger,$$

Thus:

$$\text{Tr}(\Lambda F|_S) = \text{Tr}(\Lambda F^S) + |\gamma|^2. \quad (3.21)$$

The uniform  $C^0$  bound on  $\Lambda F$  along the Yang-Mills flow and Proposition 3 gives the uniform  $C^0$  bound on  $\gamma$ .  $\square$



We now turn to the general case, and consider the HNS filtration on  $E$ :

$$0 = S^0 \subset S^1 \subset S^2 \subset \dots \subset S^p = E,$$

with stable quotients  $Q^i = S^i/S^{i-1}$ . Let  $\gamma^i$  be the second fundamental form associated to the inclusion  $S^i \subset E$ . We also consider second fundamental forms terms given by the inclusions  $S^{i-1} \subset S^i$ , which we denote by  $\gamma_{i-1}^i$ . The curvature  $F$  decomposes onto  $S^{p-1}$  and  $Q^p$  as follows:

$$F = \begin{pmatrix} F^{S^{p-1}} - \gamma^{i-1} \wedge (\gamma^{i-1})^\dagger & \tilde{\nabla} \gamma^{i-1} \\ -(\tilde{\nabla} \gamma^{i-1})^\dagger & F^{Q^p} - (\gamma^{i-1})^\dagger \wedge \gamma^{i-1} \end{pmatrix},$$

Now, because  $Q^p$  is stable, we can apply the preceding arguments from this section to get bounds on the induced curvature  $F^{Q^p}$  on  $K$  away from  $X \setminus Z_{alg}$ . By Corollary 4 we get bounds on the second fundamental form  $|\gamma^{p-1}|^2$ . The second fundamental form bounds, along with the  $C^0$  bound on  $\Lambda F$ , gives us bounds on  $\Lambda F^{S^{p-1}}$ . Now,  $S^{p-1}$  is not stable, so the arguments of Section 3 do not immediately apply. However, we can similarly decompose  $F^{S^{p-1}}$  into its components in  $S^{p-2}$  and  $Q^{p-1}$ . And because  $Q^{p-1}$  is stable, we get bounds on  $F^{Q^{p-1}}$ . Corollary 4 and the bound on  $\Lambda F^{S^{p-1}}$  give us control of the second fundamental form  $\gamma_{p-2}^{p-1}$ . Continuing in this fashion, we achieve the desired bounds on all the induced curvatures  $F^{Q^i}$  and the second fundamental form terms  $\gamma_{i-1}^i$ . Notice that  $(\gamma_{i-1}^i)^*$  is given by:

$$(\gamma_{i-1}^i)^* = \nabla^{S^i} - \nabla^{S^{i-1}} = (\nabla^E - \nabla^{S^{i-1}}) - (\nabla^E - \nabla^{S^i}) = (\gamma^{i-1})^* - (\gamma^i)^*.$$

This formula, along with the bounds on  $|\gamma_{i-1}^i|^2$  and  $|\gamma^{p-1}|^2$ , gives control of  $|\gamma^i|^2$  for all  $i$ . Of course, the constant in the above bounds depends on the compact set  $K$ . In the next section we establish a barrier function that gives precise estimates up to  $Z_{alg}$ .

## 4 A barrier function

### 4.1 The stationary case

First let us consider the case when the geometry of  $E$  is fixed. As in the previous section, we first assume the HNS filtration of  $E$  is given by  $0 \subset S \subset E$ . The induced metric  $J$  on  $S$  is a section of the coherent, torsion free sheaf  $S^* \otimes \overline{S^*}$ ; that is

$$J \in \Gamma(X, S^* \otimes \overline{S^*}),$$

and this section defines a metric on the complement of  $Z_{alg}$ . Ideally, we would like to take the function  $\sigma$  to be the norm of the determinant of  $J$  regarded as a matrix. However,  $S$  need not be a vector bundle, and so the determinant of  $J$  as a matrix is not necessarily a globally defined object. We get around this as follows. Working over  $X \setminus Z_{alg}$ ,  $S^* \otimes \overline{S^*}$  is a vector bundle (say of rank  $2r$ ), and its top wedge power  $\bigwedge^r S^* \otimes \bigwedge^r \overline{S^*}$  is a line bundle on

$X \setminus Z_{alg}$ . The determinant of the matrix  $J$  (as given in local coordinates where defined), is a section of this bundle. By Proposition (6.10) from [12], we know the reflexivization is a line bundle on all of  $X$  which is in fact isomorphic to the determinant line bundle  $\det(S^* \otimes \overline{S^*})$ . In other words we have:

$$\det(S^* \otimes \overline{S^*}) \cong \left[ \bigwedge^r S^* \otimes \bigwedge^r \overline{S^*} \right]^{**},$$

and although  $\det J \in \Gamma(X \setminus Z_{alg}, \bigwedge^r S^* \otimes \bigwedge^r \overline{S^*})$  is only defined on  $X \setminus Z_{alg}$ , we show it extends to a smooth section of  $\det(S^* \otimes \overline{S^*})$  on all of  $X$ . We accomplish this by finding a local expression which makes the extension clear. Recall the exact sequence (2.8). The induced metric  $J$  is obtained from the inclusion  $B : S \hookrightarrow E$  and the metric  $H_0$  on  $E$ . In order to find a local expression for  $J$  we need a local system of generators for the coherent, torsion free sheaf  $S$ . Fix an open set  $U \subset X$  for which we have a resolution

$$0 \longrightarrow \mathcal{O}_U^{\oplus r_\ell} \longrightarrow \mathcal{O}_U^{\oplus r_{\ell-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_U^{\oplus r_1} \xrightarrow{T} S \longrightarrow 0. \quad (4.22)$$

Here,  $T$  is given by a matrix of holomorphic functions

$$T = \begin{pmatrix} T_{1,1} & \cdots & T_{1,r_1} \\ \vdots & \ddots & \vdots \\ T_{s,1} & \cdots & T_{s,r_1} \end{pmatrix}.$$

Fix an open set  $U' \subset U$  where the first  $r$  columns of the matrix  $T$  generate  $S$ . Note that, up to permuting the columns of  $T$ , we can assume that  $\emptyset \neq U' \subset U \cap X \setminus Z_{alg}$  and that  $\overline{U} \cap Z_{alg} \neq \emptyset$ . In this case, the induced metric is given by

$$J_{\bar{q}p} = (H_0)_{\bar{\beta}\alpha} B^\alpha_\gamma T^\gamma_p \overline{B^\beta_\eta T^\eta_q}.$$

Note that  $\det(J)$  converges smoothly to zero on  $\overline{U} \cap Z_{alg}$ , since it is precisely on  $Z_{alg}$  where the rank of the composition  $B \circ T$  drops, and this is clearly independent of the resolution (4.22), or the choice of generators. We have proved the following:

**Proposition 7.** *Let  $\zeta$  be the smooth section of  $\bigwedge^r S^* \otimes \bigwedge^r \overline{S^*}$  over  $X \setminus Z_{alg}$  defined by  $\zeta = \det(J)$ . Then  $\sigma$  extends to a smooth section  $\hat{\zeta}$  of the reflexivization  $[\bigwedge^r S^* \otimes \bigwedge^r \overline{S^*}]^{**}$ . Moreover, the extension is given explicitly by*

$$\hat{\zeta}(x) = \begin{cases} \zeta(x) & \text{if } x \in X \setminus Z_{alg} \\ 0 & \text{if } x \in Z_{alg} \end{cases} \quad (4.23)$$

*In particular, we have the equality of sets  $\{\hat{\zeta} = 0\} = Z_{alg}$ .*

Of course, this same analysis carries over immediately to the case of the general filtration (2.7). In this case, we have

$$(J_{i-1})_{\bar{q}p} = (J_i)_{\bar{\beta}\alpha} (B_i)^\alpha \gamma(T_i)^\gamma \overline{(B_i)^\beta \eta(T_i)^\eta}.$$

where  $B_i$  is the map in the sequence (2.11), and  $T_i$  is the first map in the resolution of  $S^{i-1}$ . We leave the details to the reader. We now define our barrier function on  $X$ .

**Definition 6.** Fix metrics  $\phi_i$  on the line bundles  $\det(S_i^* \otimes \overline{S_i^*})$ . We then define

$$\sigma = c \prod_{i=1}^{p-1} |\det(J_i)|_{\phi_i}, \quad (4.24)$$

where  $\det(J_i)$  denotes the smooth section  $\hat{\zeta}_i$  of Proposition 7, and  $c > 0$  is chosen so that  $\max_X \sigma = 1$ .

The reader can easily verify that again in this case we have the equality of sets  $\{\sigma = 0\} = Z_{alg}$ .

## 4.2 A barrier function along the Yang-Mills flow

In the last section used the induced metrics  $J_i$  to construct a barrier function in the stationary setting. In the evolving setting, the subsheaf  $S^i$  is acted on by  $w^{S^i}(t)$  along the Yang-Mills flow, and so the induced metric evolves. We need to take care in order to ensure the barrier still provides the bounds we need. Along the flow the induced metric is given by:

$$(J_{i-1})(t)_{\bar{q}p} = (J_i)(t)_{\bar{\beta}\alpha} w^{S^i}(t)^\alpha \nu(B_i)^\nu \gamma(T_i)^\gamma \overline{w^{S^i}(t)^\beta \delta(B_i)^\delta \eta(T_i)^\eta}.$$

Of course, we could apply the above argument to construct a section  $\zeta(t)$  with the desired properties. However, the endomorphisms  $w(t)$  are uncontrolled. Moreover, for our later applications it is preferable to have a barrier function which does not depend on time. Instead, we construct a section  $\tilde{\zeta}(t)$  by taking the determinant of the metric

$$(\tilde{J}_{i-1})(t)_{\bar{q}p} = (\tilde{J}_i)(t)_{\bar{\beta}\alpha} \tilde{w}^{S^i}(t)^\alpha \nu(B_i)^\nu \gamma(T_i)^\gamma \overline{\tilde{w}^{S^i}(t)^\beta \delta(B_i)^\delta \eta(T_i)^\eta}. \quad (4.25)$$

In this case, it is straightforward to check that we have

$$\det(J_{i-1})(t) = \det(J_{i-1})(0) |\det(\tilde{w}^{S^{i-1}})|^2(t).$$

The second term on the right hand side is controlled by decomposing  $S^{i-1}$  into a direct sum of stable bundles. In particular, locally on  $X \setminus Z_{alg}$  we write

$$S^{i-1} = Q^1 \oplus Q^2 \oplus Q^3 \oplus \dots \oplus Q^{i-1},$$

where each  $Q^i$  is stable, by identifying  $Q^j$  with  $(S^{j-1})^\perp$  inside of  $S^j$ . As a result

$$\det(\tilde{w}^{S^{i-1}}) = \prod_{j=1}^{i-1} \det(\tilde{w}^{Q^j}),$$

and so by Proposition 6, there is a constant  $C > 0$  such that the following inequality holds:

$$C^{-1}|\det J(0)| \leq |\det(J)|(t) \leq C|\det(J)|(0).$$

In particular, in the evolved case we use the same  $\sigma$  constructed in the stationary case.

## 5 Estimates near the algebraic singular set

### 5.1 Estimates on the subsheaf $S$

Fix  $S^i \subset E$  one of the subsheaves appearing in the filtration (2.7). We now obtain precise estimates for the blow up rate of the curvature  $F^{S^i}$  and  $\nabla F^{S^i}$  along  $Z_{alg}$ . These estimates will depend on  $\tilde{w}^{S^i} = \tilde{w}^{S^i}(t)$  and the barrier function  $\sigma$ . In this section we work exclusively on  $S^i$ . In order to lighten the notational burden, we shall suppress the superscript  $S^i$  from all our formulae. Before beginning, we note the following elementary linear algebra lemma.

**Lemma 5.** *Let  $M$  be a  $r \times r$  hermitian positive definite matrix with eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$ . Then*

$$\mathrm{Tr}(M^{-n}) = \det(M)^{-n} \sum_{i=1}^r \prod_{j \neq i} \lambda_j,$$

where  $M^{-n}$  denotes the  $n$ -fold product of  $M^{-1}$  with itself.

As in Section 3 we shall use crucially that the complex structure induced by the gauge transformations  $w$  and  $\tilde{w}$  agree. In addition, all of the computations to follow are unchanged if the induced metric  $J$  is replaced by the normalized induced metric  $\tilde{J}$  defined by equation (4.25). With this in mind we shall not notationally distinguish between the normalized and unnormalized structure. Along the Yang-Mills flow, the evolved connection on  $S$  over  $X \setminus Z_{alg}$  is determined by the formulae:

$$A''(t) = w A''(0) w^{-1} - \bar{\partial} w w^{-1}, \quad (5.26)$$

$$A'(t) = J^{-1} \partial J - J^{-1} \overline{A''(t)} J, \quad (5.27)$$

and as we have stated this is clearly unchanged by replacing  $J(t)$  by  $\tilde{J}(t)$  and  $w(t)$  by  $\tilde{w}(t)$ . Recall that the curvature of  $S$  is the following endomorphism valued two-form over  $X \setminus Z_{alg}$ :

$$F_A(t) = \bar{\partial} A''(t) + \partial A''(t) + A'(t) \wedge A''(t) + A''(t) \wedge A'(t). \quad (5.28)$$

The main observation is that, given uniform control on the gauge transformations  $w(t)$ , the singularities of  $F_A(t)$  are due precisely to the degeneracy of the induced metric  $J$  near  $Z_{alg}$ , and thus the bounds follow on any compact subset  $K \in X \setminus Z_{alg}$ . To make this precise we compute:

$$\bar{\partial}A''(t) = \bar{\partial}wA''(0)w^{-1} + w\bar{\partial}A''(0)w^{-1} + wA''(0)\bar{\partial}w^{-1} - \bar{\partial}(\bar{\partial}ww^{-1}) \quad (5.29)$$

$$\partial A'(t) = -J^{-1}\partial JJ^{-1}\partial J - J^{-1}\partial\partial J + J^{-1}\partial JJ^{-1}\overline{A''(t)}J - J^{-1}\overline{\bar{\partial}A''(t)}J - J^{-1}\overline{A''(t)}\partial J \quad (5.30)$$

For the remainder of the estimate we work locally on an open  $U \subset X \setminus Z_{alg}$  over which  $S$  admits a trivialization. We enforce the policy that a uniform constant is independent of  $U$ . We now have:

$$\begin{aligned} |\bar{\partial}A''(t)|^2 &= \text{Tr}(g^{-1}g^{-1}\bar{\partial}A''(t)\bar{\partial}A''(t)) \\ &\leq C_1\|w\|_{C^1}^2\|w^{-1}\|_{C^1}^2(|A''(0)|^2 + |\bar{\partial}A''(0)|^2 + 1), \end{aligned}$$

for a uniform constant  $C_1$ . We now employ Lemma 2 inductively and Lemma 5 to bound  $A''(0)$ . More precisely, there are uniform constants  $C_2, C_3$  so that

$$|A''(0)|^2 \leq C_2 \prod_{j=1}^p \text{Tr}(\tilde{J}_j(0)^{-2}) \leq C_3\sigma^{-2}.$$

A similar estimate holds for  $|\bar{\partial}A''(0)|^2$  with  $\sigma^{-4}$  in place of  $\sigma^{-2}$ . In particular, there is a uniform constant  $C_3$  so that

$$|\bar{\partial}A''(t)|^2 \leq C_3\|w\|_{C^2}^2\|w^{-1}\|_{C^2}^2\sigma^{-4}.$$

A similar computation, combining equations (5.27), (5.30) and Lemma 2 shows

$$|\partial A'(t)|^2 \leq C_4\|w\|_{C^2}^2\|w^{-1}\|_{C^2}^2\sigma^{-6},$$

for a uniform constant  $C_4$ . The remaining terms on the right hand side of (5.28) are estimated in a similar fashion. Note that by equation (3.21), an estimate for  $F^S$  implies an estimate for  $|\gamma|$ . We summarize these estimates in the following two propositions.

**Proposition 8.** *There is a uniform constant  $C$  so that, on any compact set  $K \Subset X \setminus Z_{alg}$ ,*

$$|\gamma(t)|(x) + |F^S(t)|(x) \leq C\|w^S\|_{C^2(K)}\|(w^S)^{-1}\|_{C^2(K)}\sigma^{-3}(x), \quad (5.31)$$

for every  $x \in K$ .

Similar computations prove:

**Proposition 9.** *There is a uniform constant  $C$  so that, on any compact set  $K \Subset X \setminus Z_{alg}$ , we have*

$$|\nabla F^S(t)|(x) + |\bar{\nabla} F^S(t)|(x) \leq C\|w\|_{C^3(K)}\|w^{-1}\|_{C^3(K)}\sigma^{-5}(x). \quad (5.32)$$

Again, we reiterate the crucial fact that the constant  $C$  in the above propositions *does not* depend on  $K$ .

## 5.2 Estimates on the quotient sheaf $Q$

In this section we describe how to obtain curvature estimates for the quotient bundle  $Q^i = S^i/S^{i-1}$ . For the most part, these estimates follow in a similar manner as the estimates for the curvature of the bundle  $S$  considered in the previous section. Observe that local identification  $Q^i \cong \text{im}(S^{i-1})^\perp \subset S^i$  defined by  $J_i$  allows us to compute on  $(S^{i-1})^\perp$ , since the metric on  $Q^i$  is obtained precisely via this identification. Thus, using equation (2.14) we can obtain bounds for the evolved curvature  $F^{Q^i}(t)$ . Unlike the case of the subbundle  $S$ , we claim that the metric  $M_i$  on  $Q^i$  blows up along  $Z_{alg}$ , while  $M_i^{-1}$  degenerates, but remains bounded above. For example, by returning to the definition, it is clear that the induced metric  $\tilde{M}_{E/S^{i-1}}$  on  $E/S^{i-1}$  over  $X \setminus Z_{alg}$  blows up precisely where  $J_{i-1}$  degenerates, but has entries bounded below. Note that  $Q^i$  includes naturally in  $E/S^{i-1}$ , and the metric  $M^i$  is precisely the restriction of  $\tilde{M}_{E/S^{i-1}}$  to  $Q^i$ . From this, the claim follows. For this reason it is useful to write our formulas in terms of derivatives of  $M^{-1}$  rather than derivatives of  $M$ . In particular, the evolved connection is given by

$$\begin{aligned} A''(t) &= wA''(0)w^{-1} - \bar{\partial}ww^{-1} \\ A'(t) &= -M\partial M^{-1} - M^{-1}\overline{A''(t)}M. \end{aligned}$$

Since the entries of  $M_j$  are bounded below on  $X \setminus Z_{alg}$  for every  $1 \leq j \leq p$  we have the inequality

$$\det(M_i(0)) \leq C \prod_{j=i}^p \det(M_j(0)) = C \det H_0 \cdot (\det J_{i-1})^{-1}.$$

In particular, we can control  $\text{Tr}(M_j)$  above by the barrier function  $\sigma$ . Using these formulas, together with Lemma 2, equation (2.14) and the computations in Section 5.1 we easily obtain the following estimates:

**Proposition 10.** *There is a uniform constant  $C$  so that, for any compact subset  $K \Subset X \setminus Z_{alg}$ , we have*

$$|F^Q|(x) \leq C \|w^Q\|_{C^2(K)} \|(w^Q)^{-1}\|_{C^2(K)} \sigma^{-3}(x), \quad (5.33)$$

and

$$|\nabla F^Q|(x) + |\bar{\nabla} F^Q|(x) \leq C \|w^Q\|_{C^3(K)} \|(w^Q)^{-1}\|_{C^3(K)} \sigma^{-5}(x). \quad (5.34)$$

Again, we emphasize the point that in the above propositions,  $C$  is *independent* of the compact set  $K$ .

## 6 Computations for the maximum principle

### 6.1 The heat operator on $\gamma$

As stated in the introduction, all that remains is a  $C^1$  bound on all the second fundamental form terms. Let  $S$  be any subsheaf from the HNS filtration and let  $H$  evolve along the

Donaldson heat flow. Let  $|\gamma|^2 = g^{j\bar{k}} \text{Tr}(\gamma_{\bar{k}} \gamma_j^*)$ , where as before  $(\cdot)^*$  is the adjoint with respect to  $H$ . In this section we carry out the computation of  $(\partial_t - \Delta)|\gamma|^2$ , starting with the time derivative of  $\gamma_{\bar{k}}$ :

$$\partial_t(\gamma_{\bar{k}}) = -\partial_t(\nabla_{\bar{k}}\pi) = -\nabla_{\bar{k}}\dot{\pi},$$

where  $\nabla$  is the covariant derivative on the bundle  $\text{End}(E)$ . It will be useful to consider the covariant derivative on  $\text{Hom}(S^\perp, S)$ , denoted by  $\tilde{\nabla}$ . If  $L$  is in  $\text{Hom}(S^\perp, S)$ , then the two derivatives are related by:

$$\tilde{\nabla}L = \pi(\nabla L)(\mathbb{1} - \pi).$$

Note that  $\dot{\pi} \in \text{Hom}(S^\perp, S)$ . Thus  $\dot{\pi} = \pi\dot{\pi}(\mathbb{1} - \pi)$ , and so

$$\nabla_{\bar{k}}\dot{\pi} = \nabla_{\bar{k}}(\pi\dot{\pi}(\mathbb{1} - \pi)) = \tilde{\nabla}_{\bar{k}}\dot{\pi} + \nabla_{\bar{k}}\pi\dot{\pi}(\mathbb{1} - \pi) - \pi\dot{\pi}\nabla_{\bar{k}}\pi,$$

where the last two terms contain a composition of two operators in  $\text{Hom}(S^\perp, S)$ , and hence vanish. Along the Donaldson heat flow the derivative of the projection  $\pi$  is given by:

$$\dot{\pi} = \pi(h^{-1}\dot{h})(\mathbb{1} - \pi) = -\pi(\Lambda F)(\mathbb{1} - \pi) = g^{\ell\bar{m}}\tilde{\nabla}_\ell\gamma_{\bar{m}}.$$

Thus:

$$\begin{aligned}\partial_t(\gamma_{\bar{k}}) &= -\tilde{\nabla}_{\bar{k}}\dot{\pi} \\ &= g^{\ell\bar{m}}\tilde{\nabla}_{\bar{k}}\tilde{\nabla}_\ell\gamma_{\bar{m}} \\ &= g^{\ell\bar{m}}([\tilde{\nabla}_{\bar{k}}, \tilde{\nabla}_\ell]\gamma_{\bar{m}} + \tilde{\nabla}_\ell\tilde{\nabla}_{\bar{k}}\gamma_{\bar{m}}).\end{aligned}$$

Note  $\tilde{\nabla}_{\bar{k}}\gamma_{\bar{m}} = \tilde{\nabla}_{\bar{m}}\gamma_{\bar{k}}$ , since  $\gamma$  is  $\bar{\partial}$  closed. We have:

$$\partial_t(\gamma_{\bar{k}}) = g^{\ell\bar{m}}(R_{\bar{k}\ell}^{\bar{p}}\gamma_{\bar{p}} - F_{\bar{k}\ell}^S\gamma_{\bar{m}} + \gamma_{\bar{m}}F_{\bar{k}\ell}^Q) + \tilde{\Delta}\gamma_{\bar{k}}.$$

Recall the formula for the evolution of the adjoint:

$$\begin{aligned}\partial_t\gamma_j^* &= (\dot{\gamma}_{\bar{j}})^* - \gamma_j^*\Lambda F|_S + \Lambda F|_Q\gamma_j^* \\ &= g^{\ell\bar{m}}(R_{\bar{j}m}^{\bar{p}}\gamma_{\bar{p}} - F_{\bar{j}m}^S\gamma_{\bar{\ell}} + \gamma_{\bar{\ell}}F_{\bar{j}m}^Q)^* + (\tilde{\Delta}\gamma_{\bar{j}})^* - \gamma_j^*\Lambda F|_S + \Lambda F|_Q\gamma_j^*.\end{aligned}$$

Taking the derivative of  $|\gamma|^2$  gives.

$$\begin{aligned}\partial_t|\gamma|^2 &= g^{j\bar{k}}g^{\ell\bar{m}}\text{Tr}(R_{\bar{k}\ell}^{\bar{p}}\gamma_{\bar{p}}\gamma_j^* - F_{\bar{k}\ell}^S\gamma_{\bar{m}}\gamma_j^* + \gamma_{\bar{m}}F_{\bar{k}\ell}^Q\gamma_j^* + \tilde{\nabla}_\ell\tilde{\nabla}_{\bar{m}}\gamma_{\bar{k}}\gamma_j^* + \gamma_{\bar{k}}\gamma_p^*(R_{\bar{j}m}^{\bar{p}}\gamma_{\bar{\ell}})^* \\ &\quad - \gamma_{\bar{k}}\gamma_\ell^*(F_{\bar{j}m}^S)^* + \gamma_{\bar{k}}(F_{\bar{j}m}^Q)^*\gamma_\ell^* + \gamma_{\bar{k}}\tilde{\nabla}_{\bar{m}}\tilde{\nabla}_\ell\gamma_j^* - \gamma_{\bar{k}}\gamma_j^*F_{\bar{m}\ell}|_S + \gamma_{\bar{k}}F_{\bar{m}\ell}|_Q\gamma_j^*).\end{aligned}$$

Note that

$$g^{j\bar{k}}g^{\ell\bar{m}}\text{Tr}(-F_{\bar{k}\ell}^S\gamma_{\bar{m}}\gamma_j^* - \gamma_{\bar{k}}\gamma_\ell^*(F_{\bar{j}m}^S)^*) = 0,$$

since in a unitary frame for  $S$  the curvature of the induced metric is anti-self dual, so  $(F_{jm}^S)^* = -F_{mj}^S$ . Similarly

$$g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\gamma_{\bar{m}} F_{k\ell}^Q \gamma_j^* + \gamma_{\bar{k}} (F_{jm}^Q)^* \gamma_\ell^*) = 0.$$

Thus,

$$\begin{aligned} \partial_t |\gamma|^2 = & g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr} (\tilde{\nabla}_\ell \tilde{\nabla}_{\bar{m}} \gamma_{\bar{k}} \gamma_j^* + \gamma_{\bar{k}} \tilde{\nabla}_{\bar{m}} \tilde{\nabla}_\ell \gamma_j^* + R_{k\ell}^{\bar{p}} \gamma_{\bar{p}} \gamma_j^* + \gamma_{\bar{k}} \gamma_p^* (R_{jm}^{\bar{p}})^* \\ & - \gamma_{\bar{k}} \gamma_j^* F_{m\ell} |_S + \gamma_{\bar{k}} F_{m\ell} |_Q \gamma_j^*). \end{aligned}$$

Along the Donaldson heat flow  $|\Lambda F|_{C^0}$  is uniformly bounded for all time, so

$$\partial_t |\gamma|^2 \leq g^{j\bar{k}} \text{Tr}(\tilde{\Delta} \gamma_{\bar{k}} \gamma_j^* + \gamma_{\bar{k}} \tilde{\Delta} \gamma_j^*) + C |\gamma|^2.$$

Having computed  $\partial_t |\gamma|^2$ , we turn to the Laplacian terms.

$$\begin{aligned} \Delta |\gamma|^2 &= g^{j\bar{k}} \text{Tr}(\Delta \gamma_{\bar{k}} \gamma_j^* + \gamma_{\bar{k}} \Delta \gamma_j^*) + |\nabla \gamma|^2 + |\bar{\nabla} \gamma|^2 \\ &\geq g^{j\bar{k}} \text{Tr}(\Delta \gamma_{\bar{k}} \gamma_j^* + \gamma_{\bar{k}} \bar{\Delta} \gamma_j^* + \gamma_{\bar{k}} \Lambda F \gamma_j^* - \gamma_{\bar{k}} \gamma_j^* \Lambda F) + |\nabla \gamma|^2 + |\bar{\nabla} \gamma|^2 \\ &\geq g^{j\bar{k}} \text{Tr}(\Delta \gamma_{\bar{k}} \gamma_j^* + \gamma_{\bar{k}} \bar{\Delta} \gamma_j^*) - \tilde{C} |\gamma|^2 + |\nabla \gamma|^2 + |\bar{\nabla} \gamma|^2. \end{aligned}$$

We complete the argument by showing the two Laplacians,  $\Delta$  and  $\tilde{\Delta}$ , agree in this case:

$$\begin{aligned} \text{Tr}(\tilde{\Delta} \gamma_{\bar{k}} \gamma_j^*) &= g^{\ell\bar{m}} \text{Tr}(\nabla_\ell (\pi \nabla_{\bar{m}} \gamma_{\bar{k}} (I - \pi)) \gamma_j^*) \\ &= g^{\ell\bar{m}} \text{Tr}(\nabla_\ell \nabla_{\bar{m}} \gamma_{\bar{k}} \gamma_j^* - \gamma_\ell^* \nabla_{\bar{m}} \gamma_{\bar{k}} \gamma_j^* + \nabla_{\bar{m}} \gamma_{\bar{k}} \gamma_\ell^* \gamma_j^*) \\ &= \text{Tr}(\Delta \gamma_{\bar{k}} \gamma_j^*). \end{aligned}$$

The last line follows since the terms containing the composition  $\gamma^* \circ \gamma^*$  vanish, as  $\gamma^* \in \text{Hom}(S, S^\perp)$ . Similarly,  $\text{Tr}(\gamma_{\bar{k}} \bar{\Delta} \gamma_j^*) = \text{Tr}(\gamma_{\bar{k}} \tilde{\Delta} \gamma_j^*)$ .

Putting everything together we obtain:

$$(\partial_t - \Delta) |\gamma|^2 \leq C |\gamma|^2 - |\nabla \gamma|^2 - |\bar{\nabla} \gamma|^2, \quad (6.35)$$

for a constant  $C$  independent of time.

## 6.2 The heat operator on $\tilde{\nabla} \gamma$

We now turn to the computation of  $(\partial_t - \Delta) |\tilde{\nabla} \gamma|^2$ . Here we use the derivative  $\tilde{\nabla}$  on the bundle  $\text{Hom}(S^\perp, S)$ , since it is precisely the component of the curvature  $F$  we are interested in. However, note that since the two connections only differ by projections, we have

$$|\tilde{\nabla} \gamma|^2 \leq |\nabla \gamma|^2. \quad (6.36)$$



This inequality will be important in the application of the maximum principle. We now compute the evolution of  $\tilde{\nabla}\gamma$ . We know

$$\tilde{\nabla}_j \gamma_{\bar{m}} = \pi \nabla_j \gamma_{\bar{m}} (\mathbb{1} - \pi),$$

and so

$$\partial_t(\tilde{\nabla}_j \gamma_{\bar{m}}) = \dot{\pi} \nabla_j \gamma_{\bar{m}} (\mathbb{1} - \pi) + \pi [\nabla_j (h^{-1} \dot{h}), \gamma_{\bar{m}}] (\mathbb{1} - \pi) + \pi \nabla_j \dot{\gamma}_{\bar{m}} (\mathbb{1} - \pi) - \pi \nabla_j \gamma_{\bar{m}} \dot{\pi} \quad (6.37)$$

As we have seen before  $\dot{\pi} = -g^{\ell\bar{p}} \tilde{\nabla}_\ell \gamma_{\bar{p}}$ . This implies that

$$\begin{aligned} \partial_t(\tilde{\nabla}_j \gamma_{\bar{m}}) &= g^{\ell\bar{p}} (-\tilde{\nabla}_\ell \gamma_{\bar{p}} \nabla_j \gamma_{\bar{m}} - \pi \nabla_j F_{\bar{p}\ell} \pi \gamma_{\bar{m}} + \gamma_{\bar{m}} (\mathbb{1} - \pi) \nabla_j F_{\bar{p}\ell} (\mathbb{1} - \pi) \\ &\quad + \pi \nabla_j \gamma_{\bar{m}} \tilde{\nabla}_\ell \gamma_{\bar{p}} + \tilde{\nabla}_j (R_{\bar{m}\ell}^{\bar{k}} \gamma_{\bar{k}} - F_{\bar{m}\ell}^S \gamma_{\bar{p}} - \gamma_{\bar{p}} F_{\bar{m}\ell}^Q) + \tilde{\nabla}_j \tilde{\nabla}_\ell \tilde{\nabla}_{\bar{p}} \gamma_{\bar{m}}). \end{aligned}$$

We now change the order of derivatives:

$$g^{\ell\bar{p}} \tilde{\nabla}_j \tilde{\nabla}_\ell \tilde{\nabla}_{\bar{p}} \gamma_{\bar{m}} = g^{\ell\bar{p}} \tilde{\nabla}_\ell \tilde{\nabla}_j \tilde{\nabla}_{\bar{p}} \gamma_{\bar{m}} = \tilde{\Delta} \tilde{\nabla}_j \gamma_{\bar{m}} - g^{\ell\bar{p}} \tilde{\nabla}_\ell (R_{\bar{p}j}^{\bar{q}} \gamma_{\bar{q}} + F_{\bar{p}j}^S \gamma_{\bar{m}} - \gamma_{\bar{m}} F_{\bar{p}j}^Q).$$

Putting everything together we get the evolution equation:

$$\begin{aligned} \partial_t(\tilde{\nabla}_j \gamma_{\bar{m}}) &= \tilde{\Delta} \tilde{\nabla}_j \gamma_{\bar{m}} + g^{\ell\bar{p}} ([\nabla_j \gamma_{\bar{m}}, \tilde{\nabla}_\ell \gamma_{\bar{p}}] - \pi \nabla_j F_{\bar{p}\ell} \pi \gamma_{\bar{m}} + \gamma_{\bar{m}} (\mathbb{1} - \pi) \nabla_j F_{\bar{p}\ell} (\mathbb{1} - \pi) \\ &\quad + \tilde{\nabla}_j (R_{\bar{m}\ell}^{\bar{k}} \gamma_{\bar{k}} - F_{\bar{m}\ell}^S \gamma_{\bar{p}} - \gamma_{\bar{p}} F_{\bar{m}\ell}^Q) - \tilde{\nabla}_\ell (R_{\bar{p}j}^{\bar{q}} \gamma_{\bar{q}} + F_{\bar{p}j}^S \gamma_{\bar{m}} - \gamma_{\bar{m}} F_{\bar{p}j}^Q)). \end{aligned}$$

Now we turn to the evolution of the norm of this quantity:

$$\begin{aligned} \partial_t |\tilde{\nabla}\gamma|^2 &= g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\partial_t(\tilde{\nabla}_j \gamma_{\bar{m}}) (\tilde{\nabla}_k \gamma_{\bar{\ell}})^*) + g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\tilde{\nabla}_j \gamma_{\bar{m}} (\partial_t(\tilde{\nabla}_k \gamma_{\bar{\ell}}))^*) \\ &\quad - g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\tilde{\nabla}_j \gamma_{\bar{m}} (\tilde{\nabla}_k \gamma_{\bar{\ell}})^* \Lambda F|_S) + g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\tilde{\nabla}_j \gamma_{\bar{m}} \Lambda F|_Q (\tilde{\nabla}_k \gamma_{\bar{\ell}})^*). \end{aligned}$$

At this point we note that, just as in the previous section, all the terms containing  $F^S$  and  $F^Q$  (but not their derivatives) will cancel using anti-self duality. Also we have

$$|g^{\ell\bar{p}} \tilde{\nabla}_\ell \gamma_{\bar{p}}| \leq |\Lambda F| \leq C,$$

which follows from the observation  $g^{\ell\bar{p}} \tilde{\nabla}_\ell \gamma_{\bar{p}}$  is the component of  $\Lambda F$  that maps  $Q$  to  $S$ . Using the decomposition of the curvature and the connection onto  $S$  and  $Q$  we have:

$$|\pi \nabla \Lambda F \pi| + |(\mathbb{1} - \pi) \nabla \Lambda F (\mathbb{1} - \pi)| \leq C |\nabla F^S| + C |\nabla F^Q| + C |\nabla \gamma| |\gamma|. \quad (6.38)$$

Thus the fact that  $R$  is bounded in  $C^0$ , inequalities (6.38) and 6.36, and the Cauchy-Schwarz inequality imply:

$$\begin{aligned} \partial_t |\tilde{\nabla}\gamma|^2 &\leq g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\tilde{\Delta} \tilde{\nabla}_j \gamma_{\bar{m}} (\tilde{\nabla}_k \gamma_{\bar{\ell}})^*) + g^{j\bar{k}} g^{\ell\bar{m}} \text{Tr}(\tilde{\nabla}_j \gamma_{\bar{m}} (\tilde{\Delta} \tilde{\nabla}_k \gamma_{\bar{\ell}})^*) \\ &\quad + C |\tilde{\nabla}\gamma|^2 (1 + |\gamma|^2) + |\tilde{\nabla}\gamma| (|\gamma|^2 + |\gamma| |\nabla F^S| + |\gamma|^2 |\nabla F^Q|). \end{aligned}$$

We now turn to the Laplacian terms.

$$\begin{aligned}
\Delta|\tilde{\nabla}\gamma|^2 &= g^{j\bar{k}}g^{\ell\bar{m}}\text{Tr}(\Delta\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^* + \tilde{\nabla}_j\gamma_{\bar{m}}\Delta(\tilde{\nabla}_k\gamma_{\bar{\ell}})^*) + |\nabla\tilde{\nabla}\gamma|^2 + |\bar{\nabla}\tilde{\nabla}\gamma|^2. \\
&\geq g^{j\bar{k}}g^{\ell\bar{m}}\text{Tr}(\Delta\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^* + \tilde{\nabla}_j\gamma_{\bar{m}}\bar{\Delta}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^* \\
&\quad + \tilde{\nabla}_j\gamma_{\bar{m}}[\Delta F, (\tilde{\nabla}_k\gamma_{\bar{\ell}})^*] + |\nabla\tilde{\nabla}\gamma|^2 + |\bar{\nabla}\tilde{\nabla}\gamma|^2 \\
&\geq g^{j\bar{k}}g^{\ell\bar{m}}\text{Tr}(\Delta\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^* + \tilde{\nabla}_j\gamma_{\bar{m}}(\Delta\tilde{\nabla}_k\gamma_{\bar{\ell}})^*) - C|\tilde{\nabla}\gamma|^2 + |\nabla\tilde{\nabla}\gamma|^2 + |\bar{\nabla}\tilde{\nabla}\gamma|^2.
\end{aligned}$$

Once again in this computation the two Laplacians,  $\Delta$  and  $\tilde{\Delta}$ , can be interchanged. Explicitly this can be seen by:

$$\begin{aligned}
\text{Tr}(\tilde{\Delta}\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^*) &= g^{p\bar{q}}\text{Tr}(\nabla_p(\pi\nabla_{\bar{q}}\tilde{\nabla}_j\gamma_{\bar{m}}(I - \pi))(\tilde{\nabla}_k\gamma_{\bar{\ell}})^*) \\
&= g^{p\bar{q}}\text{Tr}(-\gamma_p^*\nabla_{\bar{q}}\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^* \\
&\quad + \nabla_p\nabla_{\bar{q}}\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^* + \nabla_{\bar{q}}\tilde{\nabla}_j\gamma_{\bar{m}}\gamma_p^*(\tilde{\nabla}_k\gamma_{\bar{\ell}})^*) \\
&= \text{Tr}(\Delta\tilde{\nabla}_j\gamma_{\bar{m}}(\tilde{\nabla}_k\gamma_{\bar{\ell}})^*),
\end{aligned}$$

since the two terms containing the composition of terms in  $\text{Hom}(S, S^\perp)$  vanish. Putting everything together:

$$\begin{aligned}
(\partial_t - \Delta)|\tilde{\nabla}\gamma| &\leq C|\tilde{\nabla}\gamma|^2(1 + |\gamma|^2) + |\tilde{\nabla}\gamma|(|\gamma|^2 + |\gamma||\nabla F^S| + |\gamma||\nabla F^Q|) \\
&\quad - |\nabla\tilde{\nabla}\gamma|^2 - |\bar{\nabla}\tilde{\nabla}\gamma|^2.
\end{aligned} \tag{6.39}$$

This is the key inequality we will need to carry out the maximum principle. The main idea is as follows. Combining (6.39) with (6.35), and working inside of  $K$  where we know  $|\gamma|$ ,  $|\nabla F^S|^2$  and  $|\nabla F^Q|^2$  are bounded uniformly, there exist a large constant  $A$  so that:

$$(\partial_t - \Delta)(|\tilde{\nabla}\gamma|^2 + A|\gamma|^2) \leq -|\tilde{\nabla}\gamma|^2 + C.$$

Thus at a maximum point of  $|\tilde{\nabla}\gamma|^2 + A|\gamma|^2$  the left hand side is nonnegative, which implies that  $|\tilde{\nabla}\gamma|^2$  is bounded. However, as we stated in Section 3, there is no way to show the maximum does not occur on the boundary of  $K$ . As a result, we need to use the explicit barrier function.

## 7 $C^1$ estimates for the second fundamental form

We are now ready to prove a uniform estimate for  $|\tilde{\nabla}\gamma^i|$  for all  $i$ , and in doing so, finish the proof of Theorem 2. As in previous sections, we begin with the simple case  $0 \subset S \subset E$  with second fundamental form  $\gamma$ . We first show:

**Proposition 11.** *Let  $A(t)$  be a family of connections evolving along the Yang-Mills flow. Then on  $X \setminus Z_{\text{alg}}$  we have*

$$|\tilde{\nabla}\gamma|(x, t) \leq C\sigma^{-16}(x),$$

where  $C$  is a uniform constant.

We remark that this estimate is almost certainly not sharp. We expect that the exponent on the right hand side can be improved to 4. However, this estimates suffices for our purposes.

*Proof.* We apply the maximum principle to the function  $A\sigma^k|\gamma|^2 + \sigma^{4k}|\tilde{\nabla}\gamma|^2$  for some large positive constants  $A$ , and  $k \geq 8$ . This trick of anisotropic scaling has been used before, for example in [23], where the authors first learned of it. First we need to check that the function  $A\sigma^k|\gamma|^2 + \sigma^{4k}|\tilde{\nabla}\gamma|^2$  is continuous on  $X$ , smooth on  $X \setminus Z_{alg}$ , and identically zero on  $Z_{alg}$  for any  $k \geq 8$ . Note that  $\tilde{\nabla}\gamma$  is a component of the curvature tensor, and hence there is a constant  $K(t)$  depending on time, so that  $|\tilde{\nabla}\gamma| \leq K(t)$  on  $X \setminus Z_{alg}$ . By Proposition 8,  $\sigma^k|\gamma|^2$  is smooth on  $X \setminus Z_{alg}$ , continuous on  $X$  and zero on  $Z_{alg}$  as well. Thus  $A\sigma^k|\gamma|^2 + \sigma^{4k}|\tilde{\nabla}\gamma|^2$  admits a maximum on any compact time interval. Our aim is to prove that, while  $|\tilde{\nabla}\gamma|$  may be unbounded along  $Z_{alg}$  as  $t$  approaches infinity, we have uniform bounds on compact subsets of  $X \setminus Z_{alg}$ . Using the equation (6.35), we compute

$$\begin{aligned} (\partial_t - \Delta)\sigma^k|\gamma|^2 &= \sigma^k(\partial_t - \Delta)|\gamma|^2 + 2k\sigma^{k-1}\text{Re}\langle \nabla\sigma, \nabla|\gamma|^2 \rangle \\ &\quad + (k(k-1)\sigma^{k-2}|\nabla\sigma|^2 + k\sigma^{k-1}\Delta\sigma)|\gamma|^2 \\ &\leq C_1\sigma^k|\gamma|^2 - \sigma^k|\nabla\gamma|^2 + C_1(\sigma^{k-1}|\gamma| \cdot |\nabla|\gamma|| + \sigma^{k-2}|\gamma|^2) \end{aligned}$$

We now use the Kato inequality and Young's inequality to obtain

$$(\partial_t - \Delta)\sigma^k|\gamma|^2 \leq C_2\sigma^{k-2}|\gamma|^2 - \frac{1}{2}\sigma^k|\nabla\gamma|^2. \quad (7.40)$$

Similarly, we compute

$$\begin{aligned} (\partial_t - \Delta)\sigma^{4k}|\tilde{\nabla}\gamma|^2 &= \sigma^{4k}(\partial_t - \Delta)|\tilde{\nabla}\gamma|^2 + 8k\sigma^{4k-1}\text{Re}\langle \nabla\sigma, \nabla|\tilde{\nabla}\gamma|^2 \rangle \\ &\quad + (4k(4k-1)\sigma^{4k-2}|\nabla\sigma|^2 + 4k\sigma^{4k-1}\Delta\sigma)|\tilde{\nabla}\gamma|^2 \\ &\leq \sigma^{4k}(\partial_t - \Delta)|\tilde{\nabla}\gamma|^2 + C_3\sigma^{4k-2}|\tilde{\nabla}\gamma|^2 + \frac{1}{2}\sigma^{4k}|\nabla\tilde{\nabla}\gamma|^2 + \frac{1}{2}\sigma^{4k}|\bar{\nabla}\tilde{\nabla}\gamma|^2, \end{aligned} \quad (7.41)$$

where we again used the Kato inequality and Young's inequality. We now recall equation (6.39), and apply Propositions 8, 9, and 10. This gives:

$$\begin{aligned} \sigma^{4k}(\partial_t - \Delta)|\tilde{\nabla}\gamma|^2 &\leq C_4\sigma^{4k}|\tilde{\nabla}\gamma|^2 (1 + |\gamma|^2 + |\gamma||\nabla F^S| + |\gamma||\nabla F^Q|) \\ &\quad - \sigma^{4k}|\nabla\tilde{\nabla}\gamma|^2 - \sigma^{4k}|\bar{\nabla}\tilde{\nabla}\gamma|^2 \\ &\leq C_5\sigma^{4k-16}P(w)|\tilde{\nabla}\gamma|^2 - \sigma^{4k}|\nabla\tilde{\nabla}\gamma|^2 - \sigma^{4k}|\bar{\nabla}\tilde{\nabla}\gamma|^2 \\ &\leq C_6\sigma^kP(w)|\nabla\gamma|^2 - \sigma^{4k}|\nabla\tilde{\nabla}\gamma|^2 - \sigma^{4k}|\bar{\nabla}\tilde{\nabla}\gamma|^2. \end{aligned}$$

The last line follows from the assumption that  $k \geq 8$  so that  $4k - 16 \geq k$ . Here,  $P(w)$  denotes a fixed polynomial in  $\|\tilde{w}^S\|_{C^3}$ ,  $\|(\tilde{w}^S)^{-1}\|_{C^3}$ ,  $\|\tilde{w}^Q\|_{C^3}$  and  $\|(\tilde{w}^Q)^{-1}\|_{C^3}$ . Substituting this inequality into equation (7.41), we obtain

$$(\partial_t - \Delta)\sigma^{4k}|\tilde{\nabla}\gamma|^2 \leq C_7\sigma^kP(w)|\nabla\gamma|^2. \quad (7.42)$$

Suppose now that  $S$  and  $Q$  are stable. In this case the estimates of Proposition 6 are in force, and so  $P(w) \leq C_8$  for a uniform constant  $C_8$ . Let  $A = 2C_7C_8 + 2$ . Fix a time  $T \in (0, \infty)$  and denote by  $(p, s)$  the point in  $X \setminus Z_{alg} \times [0, T]$  where  $A\sigma^k|\gamma|^2 + \sigma^{4k}|\tilde{\nabla}\gamma|^2$  achieves its maximum. We now combine equations (7.40), (7.42), and apply the maximum principle at  $(p, s)$  to get

$$\begin{aligned} 0 &\leq (\partial_t - \Delta)A\sigma^k|\gamma|^2 + \sigma^{4k}|\tilde{\nabla}\gamma|^2 \leq C_9\sigma^{k-2}|\gamma|^2 - \sigma^k|\nabla\gamma|^2 \\ &\leq C_{10} - \sigma^k|\nabla\gamma|^2, \end{aligned}$$

where in the last line we used that  $k-2 \geq 6$  so that  $\sigma^{k-2}|\gamma|^2$  is uniformly bounded above by Proposition 8. Thus, we have

$$\sigma^k|\nabla\gamma|^2(p, s) \leq C_{10}.$$

We claim that this implies the desired estimate. To see this, observe that for any  $q \in X \setminus Z_{alg}$  and  $t \in [0, T]$  we have

$$\begin{aligned} \sigma^{4k}|\tilde{\nabla}\gamma|^2(q, t) &\leq \sigma^{4k}|\nabla\gamma|^2(q, t) + A\sigma^k|\gamma|^2(q, t) \\ &\leq \sigma^{4k}|\nabla\gamma|^2(p, s) + A\sigma^k|\gamma|^2(p, s) \\ &\leq C_{11}. \end{aligned}$$

Reorganizing proves that for all  $t \in [0, T]$  and  $x \in X \setminus Z_{alg}$  we have

$$|\tilde{\nabla}\gamma|(x, t) \leq C\sigma^{-16}(x),$$

for a constant  $C$  independent of  $T$ . Since  $T$  was arbitrary, the proposition follows.  $\square$

We now turn to the general case, and consider the HNS filtration on  $E$ :

$$0 \subset S^1 \subset S^2 \subset \dots \subset S^p = E,$$

with stable quotients  $Q^i = S^i/S^{i-1}$ . Recall we defined  $\gamma^i$  to be the second fundamental form associated to the inclusion  $S^i \subset E$ . Similarly we consider second fundamental forms given by the inclusions  $S^{i-1} \subset S^i$ , which we denote by  $\gamma_{i-1}^i$ . As before,  $(\gamma_{i-1}^i)^*$  is given by:

$$(\gamma_{i-1}^i)^* = \nabla^{S^i} - \nabla^{S^{i-1}} = (\nabla^E - \nabla^{S^{i-1}}) - (\nabla^E - \nabla^{S^i}) = (\gamma^{i-1})^* - (\gamma^i)^*.$$

This formula yields the following estimates:

$$|\nabla\gamma_{i-1}^i| \leq |\nabla\gamma^i| + |\nabla\gamma^{i-1}| \quad \text{and} \quad |\nabla\nabla\gamma_{i-1}^i| \leq |\nabla\nabla\gamma^i| + |\nabla\nabla\gamma^{i-1}|. \quad (7.43)$$

Thus, to complete the proof of Theorem 2, we need to control  $|\tilde{\nabla}\gamma^i|$  for all  $i$  on compact sets  $K$  away from  $Z_{alg}$ . We apply the maximum principle to the following function:

$$f = \sum_{i=1}^{p-1} (\sigma^{4k}|\tilde{\nabla}\gamma^i|^2 + A\sigma^k|\gamma^i|^2),$$

for a universal constant  $A$ . We note that this step is similar to the maximum principle argument from Proposition 11, so we only present the details here relevant to the general case.

Recall our computation of the heat operator (6.39), which we apply for all  $i$ :

$$(\partial_t - \Delta)|\tilde{\nabla}\gamma^i| \leq C|\tilde{\nabla}\gamma^i|^2(1 + |\gamma^i|^2) + |\tilde{\nabla}\gamma^i| \left( |\gamma^i|^2 + |\gamma^i||\nabla F^{S^i}| + |\gamma^i||\nabla F^{E/S^i}| \right) - |\nabla\tilde{\nabla}\gamma^i|^2 - |\bar{\nabla}\tilde{\nabla}\gamma^i|^2. \quad (7.44)$$

Now, because  $S^i$  and  $E/S^i$  may not be stable, the induced curvatures are not bounded, so we cannot immediately carry out the argument as in Proposition 11. We can however decompose the problem terms  $|\nabla F^{S^i}|$  and  $|\nabla F^{E/S^i}|$  onto subsheaves and quotient sheaves, which we demonstrate for  $\nabla F^{S^i}$ :

$$\begin{pmatrix} \nabla^{S^{i-1}}(F^{S^{i-1}} - \gamma_{i-1}^i \wedge \gamma_{i-1}^{i*}) + \gamma_{i-1}^i \nabla \gamma_{i-1}^{i*} & \nabla \nabla \gamma_{i-1}^i + \gamma_{i-1}^i (F^{Q^i} - \gamma_{i-1}^{i*} \wedge \gamma_{i-1}^i) \\ \gamma_{i-1}^{i*} (F^{S^{i-1}} - \gamma_{i-1}^i \wedge \gamma_{i-1}^{i*}) + \nabla \nabla \gamma_{i-1}^{i*} & \gamma_{i-1}^{i*} \nabla \gamma_{i-1}^i + \nabla^{Q^i} (F^{Q^i} - \gamma_{i-1}^{i*} \wedge \gamma_{i-1}^i) \end{pmatrix}.$$

This gives the following estimate:

$$|\nabla F^{S^i}| \leq |\nabla F^{S^{i-1}}| + |\nabla F^{Q^i}| + 4|\nabla \gamma_{i-1}^i||\gamma_{i-1}^i| + |\gamma_{i-1}^i|(|F^{S^{i-1}}| + |F^{Q^i}|) + 2|\nabla \nabla \gamma_{i-1}^i| + 2|\gamma_{i-1}^i|^3.$$

We note a similar estimate exists for  $\nabla F^{E/S^i}$ . From now on, in the interest of simplicity, we will use the symbol  $\lesssim$  to denote an inequality which holds up to terms depending only on uniform constants and powers of  $\sigma$ , and suppress such terms. Applying the above estimates, combined with (7.43) and (7.44) we get:

$$\begin{aligned} (\partial_t - \Delta)|\tilde{\nabla}\gamma^i|^2 &\lesssim C(|\nabla \gamma^i|^2 + |\nabla \gamma^{i-1}|^2) + (|\nabla \gamma^i| + |\nabla \gamma^{i-1}|)(|\nabla F^{S^{i-1}}| + |\nabla F^{E/S^{i+1}}| \\ &\quad + |\nabla F^{Q^i}|) + (|\nabla \gamma^i| + |\nabla \gamma^{i-1}|)(|F^{S^{i-1}}| + |F^{E/S^{i+1}}| + |F^{Q^i}|) \\ &\quad + 2(|\nabla \gamma^i| + |\nabla \gamma^{i-1}|)(|\nabla \nabla \gamma^i| + |\nabla \nabla \gamma^{i-1}|) - |\nabla \nabla \gamma^i|^2 - |\nabla \nabla \gamma^{i-1}|^2. \end{aligned}$$

Applying Young's inequality to  $2(|\nabla \gamma^i| + |\nabla \gamma^{i-1}|)(|\nabla \nabla \gamma^i| + |\nabla \nabla \gamma^{i-1}|)$  gives, for a larger constant  $C$ :

$$\begin{aligned} (\partial_t - \Delta)|\tilde{\nabla}\gamma^i|^2 &\lesssim C(|\nabla \gamma^i|^2 + |\nabla \gamma^{i-1}|^2) + (|\nabla \gamma^i| + |\nabla \gamma^{i-1}|)(|\nabla F^{S^{i-1}}| + |\nabla F^{E/S^{i+1}}| \\ &\quad + |\nabla F^{Q^i}|) + (|\nabla \gamma^i| + |\nabla \gamma^{i-1}|)(|F^{S^{i-1}}| + |F^{E/S^{i+1}}| + |F^{Q^i}|) \\ &\quad - (1 - \epsilon_i)|\nabla \nabla \gamma^i|^2 - (1 - \epsilon_i)|\nabla \nabla \gamma^{i-1}|^2. \end{aligned}$$

Now, we have bounds on all the  $F^{Q^i}$  terms from stability. We can also continue to decompose the remaining curvature terms, and because they are all multiplied by  $|\nabla \gamma^i|$  with exponent 1, we can continue to apply Young's inequality and control the second derivative terms. Note that by this decomposition, the computation for the heat operator on  $|\tilde{\nabla}\gamma^i|$  will contain  $|\tilde{\nabla}\gamma^k|$  terms on the right hand side for  $k \neq i$ . This observation

necessitates our definition of  $f$  above. Assuming that each  $\epsilon_i$  is small so that their entire sum is less than  $1/2$ , and suppressing stable curvature terms, we have:

$$(\partial_t - \Delta) \sum_{i=1}^{p-1} |\tilde{\nabla} \gamma^i|^2 \lesssim C \sum_{i=1}^{p-1} |\nabla \gamma^i|^2 - \frac{1}{2} \sum_{i=1}^{p-1} |\nabla \nabla \gamma^i|^2.$$

Of course, up to this point we have suppressed the dependence on  $\sigma$  and the normalized complex gauge transformations that prove the curvature bounds. However, it is clear at this point that the method of Proposition 11 applies. In particular, there exists a  $k_0 > 0$  and a universal constant  $A$  so that the function

$$f = \sum_{i=1}^{p-1} (\sigma^{4k_0} |\tilde{\nabla} \gamma^i|^2 + A \sigma^{k_0} |\gamma^i|^2),$$

achieves its maximum on  $X \setminus Z_{alg}$ . Applying the maximum principle and on  $X \setminus Z_{alg} \times [0, T]$ , we obtain

**Proposition 12.** *Let  $A(t)$  be a sequence of connections evolving along the Yang-Mills flow. Then there exists a  $k_0 > 0$  and a universal constant  $C$  so that for all  $i$ :*

$$|\tilde{\nabla} \gamma^i| \leq \sigma^{-2k_0} C,$$

where  $C$  is uniform in time.

With this proposition we have succeeded in proving Theorem 2. Theorem 1 follows immediately from this. We can now give a proof of the Donaldson-Uhlenbeck-Yau theorem.

*Proof of Corollary 1.* Let  $E$  be a holomorphic vector bundle over a compact Kähler manifold. Suppose that  $E$  is indecomposable, and stable in the sense of Mumford-Takemoto. In this case  $Gr^{hns}(E) = E$ , and hence  $Z_{alg} = \emptyset$ . By Theorem 1 we have  $Z_{an} = \emptyset$ . At this juncture we point out that the main technical ingredient not proved in this paper is Simpson's lower bound for the Donaldson functional. We now apply Hong-Tian [8] to obtain that the Yang-Mills flow converges smoothly to a Yang-Mills connection on a limiting bundle  $E_\infty$  defined over all of  $X$ . By a result of the first author in [10] (which again relies only on Simpson's lower bound), we have  $E_\infty \cong Gr^{hns}(E)^{**} = E^{**} = E$ . Thus,  $E_\infty$  is isomorphic to  $E$ . Since  $E$  is stable, the limiting connection is Hermitian-Einstein. The reverse implication is classical, and so we omit it; see for instance [12].  $\square$

The proof of Corollary 2 is identical to the second argument in the proof of Corollary 1. We leave the details to the reader.

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